Homework #4

Due: Mon., Feb. 20^{th}, 2023

Unless otherwise specified, V is a vector space and L is a Lie algebra over a field $\mathbb{F}$.

#1 Centrally Direct Let L_1 and L_2 be subalgebras of L where $L = L_1 \oplus L_2$ (a Lie algebra direct sum): In other words, for every $x \in L$ there exists unique $x_1 \in L_1$ and $x_2 \in L_2$ such that $x = x_1 + x_2$. Moreover, given any $y \in L_1$ and $z \in L_2$, we have $[x.y] = 0$. Prove that $Z(L) = Z(L_1) \oplus Z(L_2)$.

#2 Directly Central Let L be a Lie algebra and $I \lhd L$. Next, let $J = C_L(I) = \{x \in L \mid [x, y] = 0 \text{ for all } y \in I\}$ (i.e. J is the centralizer of I in L).

Note: From a previous homework we know $J = C_L(I)$ is an ideal of L since I is an ideal of L.

(a) Suppose that all derivations of I are inner (i.e. if $\partial : I \to I$ is a derivation, then there exists some $x \in I$ such that $\partial = \mathrm{ad}_x$). Show that $L = I + J$.

(b) If in addition, $Z(I) = \{0\}$, show that $L = I \oplus J$.

#3 Derived Question Recall that $L^{(k+1)} = [L^{(k)}, L^{(k)}]$ and $L^{(0)} = L$. Let $\varphi : L_1 \to L_2$ be an **epi**morphism.

Show that $\varphi\left(L_1^{(k)}\right) = L_2^{(k)}$ for all k.

#4 Adjoint Representation Recall that $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is defined by $\mathrm{ad}(x) = \mathrm{ad}_x$ where $\mathrm{ad}_x(y) = [x, y]$ for all $x, y \in L$. We already know that $\mathrm{ad}_x \in \mathfrak{gl}(L)$ for all $x \in L$ (i.e. ad_x is a linear operator – in fact, $\mathrm{ad}_x \in \mathrm{Der}(L)$). Show that ad is a homomorphism. Then show that $L/Z(L) \cong \mathrm{ad}(L)$.