Math 4010/5530 **Homework #5** **Due: Mon., Mar. 27th, 2023**

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ and let V be a $\mathfrak{g}$-module.

#1 Submodule Basics A warm up.

(a) Let W_1 and W_2 be submodules of V. Show that $W_1 \cap W_2$ and $W_1 + W_2$ are submodules. Is $W_1 \cup W_2$ a submodule? Discuss.

(b) Let $\varphi : V \to W$ be a $\mathfrak{g}$-module homomorphism. Also, let U be a submodule of W. Show that the inverse image of U, $\varphi^{-1}(U) = \{\mathbf{v} \in V \mid \varphi(\mathbf{v}) \in U\}$, is a submodule of V which contains the kernel.

#2 ~~Meth~~ Math Lab Let $\mathfrak{h} = \text{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ be the Heisenberg Lie algebra (over $\mathbb{C}$) whose bracket structure is defined by: $[\mathbf{x}, \mathbf{y}] = \mathbf{z}$ and $\mathbf{z}$ is central (i.e., $[\mathbf{x}, \mathbf{z}] = [\mathbf{y}, \mathbf{z}] = 0$). Let $\mathbb{C}[t]$ be the algebra of polynomials with complex coefficients (and indeterminate t). For $f(t) \in \mathbb{C}[t]$ define $\mathbf{x} \bullet f(t) = f'(t)$ (differentiation), $\mathbf{y} \bullet f(t) = tf(t)$ (multiplication by t), $\mathbf{z} \bullet f(t) = f(t)$ (multiplication by 1), and extend linearly. Assuming bilinearity of this action, show $\mathbb{C}[t]$ is indeed a $\mathfrak{h}$-module. Is this an irreducible module?

#3 Irreducible Let $V \neq \{0\}$. Show V is irreducible if and only if V is generated by any of its non-zero elements (i.e., for any $0 \neq \mathbf{v} \in V$ we have that the submodule generated by $\mathbf{v}$ is all of V).

Note: The submodule generated by $\mathbf{v}$ is the span of all elements of the form:

$$\mathbf{x}_1 \bullet (\mathbf{x}_2 \bullet (\cdots (\mathbf{x}_\ell \bullet \mathbf{v}) \cdots)) \qquad \text{where} \quad \mathbf{x}_1, \dots, \mathbf{x}_\ell \in \mathfrak{g} \quad \text{and} \quad \ell \geq 0.$$

By the way, $\ell = 0$ gives us the empty product (no elements acting on $\mathbf{v}$) which is just $\mathbf{v}$ itself.