Homework #6

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ and let V be a $\mathfrak{g}$-module.

Also, let $\mathfrak{sl}_2(\mathbb{C}) = \text{span}\{f, h, e\}$ where as usual: $[h, f] = -2f$, $[h, e] = 2e$, and $[e, f] = h$.

#1 Restriction Given a $\mathfrak{g}$-module V and a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we can think of V as a $\mathfrak{h}$-module by restricting $\mathfrak{g}$'s action to $\mathfrak{h}$: $\quad h \bullet v$ where $h \in \mathfrak{h}$ and $v \in V$.

Notice that the matrices of the form $\begin{bmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & 0 \end{bmatrix}$ sitting inside $\mathfrak{sl}_3(\mathbb{C})$ give us an isomorphic copy of $\mathfrak{sl}_2(\mathbb{C})$ (similarly we can treat $\mathfrak{sl}_k(\mathbb{C})$ as a subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ for any $k \leq n$). This allows us to treat $V = \mathfrak{sl}_3(\mathbb{C})$ as an $\mathfrak{sl}_2(\mathbb{C})$-module with action: $\mathbf{x} \bullet \mathbf{y} = [\mathbf{x}, \mathbf{y}]$ (we are restricting $\mathfrak{sl}_3$'s adjoint action to this subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$). By Weyl's theorem, we know that V is completely reducible.

Show that $V \cong V(2) \oplus V(1) \oplus V(1) \oplus V(0)$ by finding a basis of h-eigenvectors ($h = \text{diag}(1, -1, 0)$).

[*Note:* $V(m)$ denotes the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of highest weight m whose dimension is $m + 1$.]

#2 Eigen-Determinism Suppose that V is a finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module. Using Weyl's theorem and the classification of irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules, show that V is the direct sum of k irreducible submodules where $k = \dim(V_0) + \dim(V_1)$ and $V_\lambda = \{\mathbf{v} \in V \mid h \bullet \mathbf{v} = \lambda \mathbf{v}\}$.

Explain why V is **not** determined by the set of h's eigenvalues alone. However, it is determined (up to module isomorphism) by h's eigenvalues *counting multiplicity.*

#3 Not Quite Trivial We explore 1-dimensional representations.

(a) Suppose that $\varphi : \mathfrak{g} \to \mathfrak{gl}(1, \mathbb{F})$ is a 1-dimensional representation of $\mathfrak{g}$. Show that $\varphi(\mathfrak{g}') = \{\mathbf{0}\}$.
[Recall that $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.]

(b) **(Grad Students)** Show that any representation of $\mathfrak{g}/\mathfrak{g}'$ can be viewed as a representation of $\mathfrak{g}$ on which $\mathfrak{g}'$ acts trivially.

(c) **(Grad Students)** When $\mathbb{F} = \mathbb{C}$, show that if $\mathfrak{g}' \neq \mathfrak{g}$, then $\mathfrak{g}$ has infinitely many non-isomorphic 1-dimensional modules, but if $\mathfrak{g}' = \mathfrak{g}$, then the only 1-dimensional representation of $\mathfrak{g}$ is the trivial representation.

Note: For 3(a) it may be worth noting that we can identify $\mathfrak{gl}(1, \mathbb{F}) = \mathbb{F}$ (these are naturally isomorphic). Having done so, in part (a) we are showing that if φ is a 1-dimensional representation (so $\varphi \in \mathfrak{g}^*$ and it is a homomorphism), then $\varphi(\mathfrak{g}') = \mathbf{0}$. It might be helpful to note that the converse is also true: if φ is a dual vector vanishing on $\mathfrak{g}'$, then φ is a 1-dimensional representation (if we identify $\mathfrak{gl}(1, \mathbb{F})$ and $\mathbb{F}$).