Math 4010-101 **Homework #7** **Due: Mon., Apr. 17^{th}, 2023**

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$.

#1 Dualing Problem: Consider $\beta = \{(1,1,0),(2,1,0),(1,-1,1)\}$ (this is a basis for $\mathbb{R}^3$).

(a) Find the dual basis β^*. [Give formulas for each dual vector. For example: $f(x,y,z) = 2x+5y-z$. *Unnecessary note:* f is **not** one of the elements of β^*.]

(b) Explain why $f(x,y,z) = x^2$ *is not* in $(\mathbb{R}^3)^*$.

(c) Explain why $f(x,y,z) = x$ *is* in $(\mathbb{R}^3)^*$ and compute $[f]_{\beta^*}$ (this is f's β^*-coordinate vector).

(d) Find the change of basis matrices from $\text{std}^* = \{\mathbf{i}^*, \mathbf{j}^*, \mathbf{k}^*\}$ (the standard dual basis) to β^* (i.e. $[I]^{\beta^*}_{\text{std}^*}$). Also, find the change of basis matrices from to β^* to std^* (i.e. $[I]^{\text{std}^*}_{\beta^*}$).

(e) What is the relationship between $[I]^{\beta^*}_{\text{std}^*}$ and $[I]^{\beta}_{\text{std}}$?
Given two arbitrary bases α and γ for an arbitrary finite dimensional vector space (over some field $\mathbb{F}$), conjecture a relationship between $[I]^{\gamma}_{\alpha}$ and $[I]^{\gamma^*}_{\alpha^*}$.

(f) [**Grad Students**] Prove your conjecture from part (e).

#2 Look at me, Mom. I'm a module too! Let V and W be $\mathfrak{g}$-modules.

(a) The space of linear maps from V to W can be turned into a $\mathfrak{g}$-module as follows:
Let $T \in \text{Hom}(V,W) = \{S : V \to W \mid S \text{ is linear}\}$ and $g \in \mathfrak{g}$. Then for all $\mathbf{v} \in V$, define

$$(x \bullet T)(\mathbf{v}) = x \bullet T(\mathbf{v}) - T(x \bullet \mathbf{v}).$$

First, show $x \bullet T \in \text{Hom}(V,W)$. Then show that this turns $\text{Hom}(V,W)$ into a $\mathfrak{g}$-module.

Note: I did some of this in a previous class. Prove the linear/bilinear bits. I will handle the Jacobi/commutator (main) axiom for modules:

Suppose $x,y \in \mathfrak{g}$ and $T \in \text{Hom}(V,W)$. We need to show that: $x \bullet (y \bullet T) - y \bullet (x \bullet T) = [x,y] \bullet T$. This calculation is unnecessarily opaque when done with module action notation, so we switch to representation notation (I'll use square brackets for plugging stuff into a map): Say $\varphi(x)[\mathbf{v}] = x \bullet \mathbf{v}$ (this is V's action) and $\psi(x)[\mathbf{w}] = x \bullet \mathbf{w}$ (this is W's action). Then $(x \bullet T)[\mathbf{v}] = x \bullet T[\mathbf{v}] - T[x \bullet \mathbf{v}] = (\psi(x) \circ T)[\mathbf{v}] - (T \circ \varphi(x))[\mathbf{v}]$. In other words, $x \bullet T = \psi(x) \circ T - T \circ \varphi(x)$.

$$\begin{aligned} x \bullet (y \bullet T) - y \bullet (x \bullet T) &= x \bullet (\psi(y) \circ T - T \circ \varphi(y)) - y \bullet (\psi(x) \circ T - T \circ \varphi(x)) \\ &= x \bullet (\psi(y) \circ T) - x \bullet (T \circ \varphi(y)) - y \bullet (\psi(x) \circ T) + y \bullet (T \circ \varphi(x)) \\ &= \psi(x) \circ (\psi(y) \circ T) - (\psi(y) \circ T) \circ \varphi(x) - \psi(x) \circ (T \circ \varphi(y)) + (T \circ \varphi(y)) \circ \varphi(x) \\ &\quad - \psi(y) \circ (\psi(x) \circ T) + (\psi(x) \circ T) \circ \varphi(y) + \psi(y) \circ (T \circ \varphi(x)) - (T \circ \varphi(x)) \circ \varphi(y) \\ &= (\psi(x) \circ \psi(y) - \psi(y) \circ \psi(x)) \circ T - T \circ (\varphi(x) \circ \varphi(y) - \varphi(y) \circ \varphi(x)) \\ &= [\psi(x), \psi(y)] \circ T - T \circ [\varphi(x), \varphi(y)] = \psi([x,y]) \circ T - T \circ \varphi([x,y]) = [x,y] \bullet T \end{aligned}$$

In the above calculation, recall that function composition is associative (so many parentheses could be dropped). Also, note how the red terms (and then blue terms) cancel each other out. The first equality in the last line follows from noting that the brackets in $\mathfrak{gl}(V)$ and $\mathfrak{gl}(W)$ (i.e., the codomains of φ and ψ) are commutator brackets. Then the final two equalities follow from the fact that representations are homomorphisms and from the definition of our action.

(b) We can turn $\mathbb{F}$ into a $\mathfrak{g}$-module via the trivial action: $x \bullet s = 0$ for all $s \in \mathbb{F}$. Explain how this then allows us to define a dual module V^* for any $\mathfrak{g}$-module V [*Hint:* Use part (a).] What is the action of $\mathfrak{g}$ on V^*?

#3 Like Mt. Fuji Reflected in a Lake: Recall that $V(m)$ is the $\mathfrak{sl}_2(\mathbb{C})$ module with highest weight m (where m is a non-negative integer).

(a) Prove that $V(1)^* \cong V(1)$.

(b) [**Grad Students**] Prove $V(m)^* \cong V(m)$.

(c) [**Everyone**] Assuming the grad student problem (part (c)) and assuming the fact that as $\mathfrak{g}$-modules:

$$\left(\bigoplus_{i=1}^{\ell} W_i\right)^* \cong \bigoplus_{i=1}^{\ell} W_i^*$$

for any finite dimensional $\mathfrak{g}$-modules W_i, prove that $V^* \cong V$ for any finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module V.

Slogan: $\mathfrak{sl}_2$-modules are self dual!

For Completeness, a sketchy proof of our direct sum fact (in an even more general context): Let $W_1, \ldots, W_\ell$ be $\mathfrak{g}$-modules for some Lie algebra $\mathfrak{g}$ (over $\mathbb{F}$). For each $j = 1, \ldots, \ell$, let $\pi_j : \oplus_i W_i \to W_j$ be the projection onto the j^{th} component. Thus if $\mathbf{w} = (w_1, \ldots, w_\ell) \in \oplus_i W_i$ where $w_k \in W_k$ for all $k = 1, \ldots, \ell$, then $\pi_j(\mathbf{w}) = w_j$. It is easy to show that π_j is a $\mathfrak{g}$-module homomorphism.
Let $(f_1, \ldots, f_\ell) \in \oplus_i W_i^*$ where $f_k \in W_k^*$ for all $k = 1, \ldots, \ell$. Since compositions and sums of linear maps are still linear, $\mathbf{f} = f_1 \circ \pi_1 + \cdots + f_\ell \circ \pi_\ell$ is linear. Notice that $\mathbf{f}(w_1, \ldots, w_\ell) = (f_1 \circ \pi_1)(w_1, \ldots, w_\ell) + \cdots + (f_\ell \circ \pi_\ell)(w_1, \ldots, w_\ell) = f_1(w_1) + \cdots + f_\ell(w_\ell)$ so that $f : (\oplus_i W_i) \to \mathbb{F}$ (i.e., $f \in (\oplus_i W_i)^*$. Therefore, $\varphi(f_1, \ldots, f_\ell) = f_1 \circ \pi_1 + \cdots + f_\ell \circ \pi_\ell$ defines a map $\varphi : \oplus_i W_i^* \to (\oplus_i W_i)^*$.
It is not hard to show that φ is itself a linear map: $\varphi(\mathbf{f} + \mathbf{g}) = \varphi(\mathbf{f}) + \varphi(\mathbf{g})$ and $\varphi(c\mathbf{f}) = c\varphi(\mathbf{f})$.
Given $x \in \mathfrak{g}$, $\mathbf{f} \in \oplus_i W_i^*$, and $(w_1, \ldots, w_\ell) \in \oplus_i W_i$,

$$\begin{aligned}
&\varphi(x \bullet \mathbf{f})[(w_1, \ldots, w_\ell)] = \varphi(x \bullet f_1, \ldots, x \bullet f_n)[(w_1, \ldots, w_\ell)] \\
&\quad = ((x \bullet f_1) \circ \pi_1)[(w_1, \ldots, w_\ell)] + \cdots + ((x \bullet f_\ell) \circ \pi_\ell)[(w_1, \ldots, w_\ell)] \\
&\quad = (x \bullet f_1)[w_1] + \cdots + (x \bullet f_\ell)[w_\ell] = -f_1(x \bullet w_1) - \cdots - f_\ell(x \bullet w_\ell) \\
&\quad = -(f_1 \circ \pi_1)[(x \bullet w_1, \ldots, x \bullet w_\ell)] - \cdots - (f_\ell \circ \pi_\ell)[(-x \bullet w_1, \ldots, -x \bullet w_\ell)] \\
&\quad = -\varphi(\mathbf{f})[x \bullet (w_1, \ldots, w_\ell)] = (x \bullet \varphi(\mathbf{f}))[(w_1, \ldots, w_\ell)].
\end{aligned}$$

Therefore, $\varphi(x \bullet \mathbf{f}) = x \bullet \varphi(\mathbf{f})$ (i.e., φ is a module homomorphism).
Next, consider (the linear) maps $\iota_j : W_j \to \oplus_i W_i$ defined by $\iota_j(w) = (0, \ldots, 0, w, 0, \ldots, 0)$ where the w is in the j^{th} slot. Notice that $(\pi_j \circ \iota_j)[w] = w$. If we define $\psi : (\oplus_i W_i)^* \to \oplus_i W_i^*$ by $\psi(f) = (f \circ \iota_1, \ldots, f \circ \iota_\ell)$, then it is not hard to see that ψ is the inverse of φ. Thus φ is a module isomorphism.