Marius Sophus Lie (1842–1899)

"I think that a mathematician is well suited to be in prison."

–Sophus Lie

It was mid-August of 1870 . The day was one of those restless days of the first month of the Franco-Prussian war when people were distrustful. The French police arrested a suspicious-looking young man who was wandering in lonely places in the forest of Fontainebleau (near Paris) stopping now and then to make notes and drawings in his notebook. "He was of tall stature and had the classic Nordic appearance. A full blond beard framed his face and his grey-blue eyes sparkled behind his eyeglasses. He gave the impression of unusual physical strength" (E. Cartan). The police searched him and found a map, letters in German and papers full of mysterious formulas, complexes, diagrams and names. He was suspected of being a German spy and imprisoned.

The arrested man was Sophus Lie, a young Norwegian mathematician, visiting the mathematical capitals of Europe. He had received a travel grant from the University of Christiania for his first mathematical paper (1869).

Sophus Lie had to stay in prison in Fontainebleau for 4 weeks before his French colleague Gaston Darboux learned about the incident and arrived on behalf of the Academy of Sciences with a release order signed by the Minister of Home Affairs.

Lie himself had taken things truly philosophically and made good use of his time in prison. For, as he recounted later, in these forced leisure days he had plenty of peace and quiet to concentrate on his problems and advance them essentially. In a letter to his Norwegian friend Ernst Motzfeldt written directly after the release, Sophus Lie remarked: "I think that a mathematician is well suited to be in prison."

[Prologue from "Elementary Lie Group Analysis and Ordinary Differential Equations" by Nail H. Ibragimov]

Infinite-Dimensional Lie Algebras

by Minoru Wakimoto

An elegant shape of the left half of Mt. Fuji reflected in the surface of a lake, this is the proportion of the finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{C})$. In this figure, ◯ is an element of V, and they all together form a basis of V. They are eigenvectors of h, and the height of each ◯ expresses its eigenvalue. The arrow ↑ shows the action of e and the arrow ↓ shows the action of f. Each column spans an irreducible invariant subspace, and each row spans an eigenspace of h. In Figure 1.1, V contains m_1 irreducible representations of dimension 1 (trivial representation), m_2 of dimension 2, m_3 of dimension 3, and so on, and V is the direct sum of these irreducible components.

Note. There are some facts that can be obviously read off from Figure 1.1. For example, let V_j be the eigenspace of V with eigenvalue j with respect to the action of the operator h. One can see that

1. $\dim V_j = \dim V_{-j}$,
2. the sequences $\{\dim V_{2j}\}_{j=0,1,2,\cdots}$ and $\{\dim V_{2j+1}\}_{j=0,1,2,\cdots}$ are both monotone decreasing,
3. $\dim V^e = \dim V_0 + \dim V_1$, where $V^e := \{v \in V; ev = 0\}$ is the vector subspace spanned by singular vectors.

etc. □

$\mathfrak{sl}(2,\mathbb{C})$ is a tiny Lie algebra of dimension only 3, and (1.11) describes all of its structure as a Lie algebra. It has countably many irreducible finite-dimensional representations, and they are classified by the dimension of its representation space. Namely, for each natural number n, there exists, up to equivalence, only one n-dimensional irreducible representation, and the action of e, f, h on its representation space can be described by (1.18) by letting $m = n - 1$. Finite-dimensional representations can be decomposed into the direct sum of irreducible invariant subspaces as above. Even when a representation space is of infinite dimension and contains a finite-dimensional invariant subspace, it similarly decomposes into the direct sum of irreducible components.

A Lie algebra $\mathfrak{g}$ itself is nothing but an algebraic system with a bracket operation. But its representation (π, V) is very active. $\pi(x)$, the element that is an element x of the Lie algebra $\mathfrak{g}$ with the cover π, is an element of $\mathrm{End}(V)$. Namely, it is a linear operator on V. The linear operator acts directly on any elements of V, and we let them move here and there.

For example, in the case of $\mathfrak{sl}(2,\mathbb{C})$, as one can see from Figure 1.1, $\pi(e)$ plays the role of a pump sending ◯ one floor above rhythmically, and $\pi(f)$ is a pump sending one floor below. Watching this

figure, one might hear a beat of ◯ moving up and down according to the work of the pumps $\pi(e)$ and $\pi(f)$. Moreover, these ◯ utter their eigenvalues in reply to the action of $\pi(h)$. When one says a 'representation', elements of the Lie algebra fit perfectly with elements of the representation, and they form a community, even though it is small.

Even if the Lie algebra under consideration is not $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{sl}(2,\mathbb{C})$ sometimes stays nearby:

DEFINITION 1.46. Let $\mathfrak{g}$ be a Lie algebra. If elements h, e, f of $\mathfrak{g}$ satisfy the relation (1.11), this triple (h, e, f) is called an **S-triple**.

The vector subspace spanned by an **S-triple** (h, e, f) is a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. In particular, if $\mathfrak{g}$ is a (finite- or infinite-dimensional) simple Lie algebra, there are many S-triples inside $\mathfrak{g}$, and an S-triple chosen properly plays quite an important role in studying the structure and the representations of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ itself is a representation space of an S-triple via its adjoint action. In the next section, we will study the structure of simple finite-dimensional Lie algebras. The only tools used there are S-triples and an invariant bilinear form (the Killing form).

Since ancient times, it has been the charm of music that has soothed the fiercest *warriors* (or *samurai*). This law seems to be universal in the physical universe, and it is also true in the world of Lie algebras. Although the theme of this book is infinite-dimensional Lie algebras, the music that plays an extremely important role in analyzing their structures and their representations is this $\mathfrak{sl}(2,\mathbb{C})$, the smallest and the loveliest algebra among the simple finite-dimensional Lie algebras. Chapter 2 of this book is the main part of representation theory of BKM algebras and BKM superalgebras, and an S-triple, i.e. $\mathfrak{sl}(2,\mathbb{C})$, always appears in an important scene. "BRING ME TO YOUR MIND!"- under this tender melody, obstructions facing us easily tumble down and the theory hidden in a veil of infinite-dimensional Lie algebras will be gradually revealed. $\mathfrak{sl}(2,\mathbb{C})$ is wonderful! Chapter 2 of this book is a paean to $\mathfrak{sl}(2,\mathbb{C})$, and the analysis of representations and the structure of infinite-dimensional Lie algebras given there is just one application of the representation theory of $\mathfrak{sl}(2,\mathbb{C})$.

1.3. Structure of Simple Finite-Dimensional Lie Algebras

In this section, let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, and let us study its structure. An important role in studying representations and the structures of $\mathfrak{g}$ is played by a Cartan subalgebra.