Math 4710/5710-101 # Homework #2 **Due: Wed., Sept. 3^{rd}, 2014**

TOTAL ORDERS: A set T equipped with a relation "$<$" is **totally ordered** if "$<$" has the following properties:

Comparability For all $a, b \in T$, either $a < b$, $b < a$, or $a = b$.

Non-reflexive For all $a \in T$, $a \not< a$.

Transitive For all $a, b, c \in T$, if $a < b$ and $b < c$, then $a < c$.

Notes: A relation which satisfies the non-reflexive and transitive properties (but not necessarily comparability) is called a *strict partial order*. Next, $b > a$ is the same as $a < b$. Also, $a \leq b$ means either $a < b$ or $a = b$. Likewise, $a \geq b$ means either $b < a$ or $a = b$.

MORE DEFINITIONS: Let T be a totally ordered set with order relation $<$. Let $A \subseteq T$.

Lower Bound $x \in T$ is a **lower bound** for A if $x \leq a$ for all $a \in A$.

Upper Bound $x \in T$ is an **upper bound** for A if $x \geq a$ for all $a \in A$.

Infimum $x \in T$ is a **greatest lower bound** or **infimum** for A if $x \leq a$ for all $a \in A$ (x is a lower bound) and for any other lower bound $y \in T$ we have $y \leq x$ (x is the largest possible lower bound).

Supremum $x \in T$ is a **least upper bound** or **supremum** for A if $x \geq a$ for all $a \in A$ (x is an upper bound) and for any other upper bound $y \in T$ we have $y \geq x$ (x is the smallest possible lower bound).

Least Upper Bound Property An ordered set has the *least upper bound property* if every non-empty subset that is bounded above has a supremum.

Greatest Lowest Bound Property An ordered set has the *greatest lower bound property* if every non-empty subset that is bounded below has an infimum.

PROBLEMS – HEY, WE'VE ALL GOT 'EM: Let T be a totally ordered set with order relation $<$. Let $A \subseteq T$.

1. Suppose A has an infimum. Show that it's unique (so we can say *the* infimum and denote it $\inf(A)$).
2. Suppose A has a supremum. Show that it's unique (so we can say *the* supremum and denote it $\sup(A)$).
3. Give examples of sets with (a) both supremum and infimum, (b) no supremum or infimum, (c) a supremum but no infimum, and (d) an infimum but no supremum.
4. [**Grad Problem**] Show that the least upper bound property implies the greatest lower bound property. [The converse is true as well and has essentially the same proof with all of the $<$'s flipped to $>$'s.]