**Math 4710/5710-101** | # Homework #9 | **Due: Fri., Dec. $5^{\text{th}}$, 2014 by 4pm**

Recall that a **net** in a topological space $X$ is a function from a **directed set** $J$ (i.e. $J$ is partially ordered by $\leq$ and for every $i, j \in J$ there exists some $k \in J$ such that $i \leq k$ and $j \leq k$) to $X$. Moreover, we say that a net $(x_j)_{j\in J}$ in $X$ **converges** to $x \in X$ if for every neighborhood $U$ of $x$ there exists some $N \in J$ such that $x_j \in U$ for all $j \geq N$. This is denoted: $x_j \to x$.

Also, recall that a function $f : X \to Y$ (where $X$ and $Y$ are topological spaces) is said to be **continuous at** $x \in X$ if for every neighborhood $U$ of $f(x)$ there exists some neighborhood $V$ of $x$ such that $V \subseteq f^{-1}(U)$.

**Problem #1(a):** Let $f : X \to Y$ be continuous at $x \in X$ and let $(x_j)_{j\in J}$ be a net in $X$ such that $x_j \to x$. Show that the net $(f(x_j))_{j\in J}$ converges to $f(x)$.

**Grad. Problem #1(b):** Suppose that for any net $(x_j)_{j\in J}$ such that $x_j \to x$, we have $f(x_j) \to f(x)$. Show that $f$ is continuous at $x$.

*Hint:* Suppose that $f$ is not continuous at $x$. Build a net which converges to $x$ but whose image does not converge to $f(x)$. You may find the proof that "$x$ belongs to the closure of a set $A$ if and only if there is a net in $A$ converging to $x$" helpful/instructive here.

Recall that $\mathcal{F} \subseteq \mathcal{P}(X)$ is a **filter** in $X$ if $\mathcal{F} \neq \varnothing$, $\varnothing \notin \mathcal{F}$, $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, and $A \subseteq B$ where $A \in \mathcal{F}$ implies $B \in \mathcal{F}$. Morever, $\mathcal{F}$ is said to be an **ultrafilter** if $\mathcal{F}$ is a maximal filter (i.e. it is not properly contained in any other filter). This is equivalent to the condition that for any $A \subseteq X$ we have $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.

**Problem #2:** Let $\mathcal{F} = \{A \subseteq X \mid X - A \text{ is finite}\}$. Show that $\mathcal{F}$ is a filter if and only if $X$ is infinite. Moreover, in the case that $\mathcal{F}$ is a filter (i.e. $X$ is infinite), explain why $\mathcal{F}$ is never an ultrafilter.

Recall that a topological space $X$ is called **completely regular** if for every closed set $A \subseteq X$ and every point $x \in X$ such that $x \notin A$ there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(a) = 1$ for all $a \in A$ (i.e. closed sets and point can be separated by continuous functions).

**Problem #3:** Let $X$ be a completely regular space. Let $A, B \subseteq X$ be disjoint closed sets and assume $A$ is compact. Prove there exists a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

*Comment:* This falls under our philosophy that compact sets behave like points.

*Hint:* Since $X$ is completely regular, we can separate each point $a \in A$ from the closed set $B$ with a continuous function $f_a$. Notice that the inverse image of the open interval $(-1/2, 1/2)$ under the map $f_a$ is an open set containing $a$. This should give you an open cover of $A$. Averaging some finite collection of functions should give you your desired function.