Math 4710/5710 # Homework #10 **Due: Wed., Apr. 30th, 2025**

Recall that a **net** in a topological space X is a function from a **directed set** J (i.e. J is partially ordered by $\leq$ and for every $i, j \in J$ there exists some $k \in J$ such that $i \leq k$ and $j \leq k$) to X. Moreover, we say that a net $(x_j)_{j\in J}$ in X **converges** to $x \in X$ if for every neighborhood U of x there exists some $N \in J$ such that $x_j \in U$ for all $j \geq N$. This is denoted: $x_j \to x$.

Also, recall that a function $f : X \to Y$ (where X and Y are topological spaces) is said to be **continuous at** $x \in X$ if for every neighborhood U of $f(x)$ there exists some neighborhood V of x such that $V \subseteq f^{-1}(U)$.

#1 Continuity via Nets: Let X and Y be topological spaces.

(a) Let $f : X \to Y$ be continuous at $x \in X$ and let $(x_j)_{j\in J}$ be a net in X such that $x_j \to x$.
Show the net $(f(x_j))_{j\in J}$ converges to $f(x)$.

(b) **[Grad.]** Suppose that for any net $(x_j)_{j\in J}$ such that $x_j \to x$, we have $f(x_j) \to f(x)$.
Show f is continuous at x.

Hint: Suppose that f is not continuous at x. Build a net which converges to x but whose image does not converge to $f(x)$. You may find the proof that "x belongs to the closure of a set A if and only if there is a net in A converging to x" helpful/instructive here.

Putting these results together, we have:

f is continuous at x if and only if for every net such that $x_i \to x$, we have $f(x_i) \to f(x)$.

Recall that $\mathcal{F} \subseteq \mathcal{P}(X)$ is a **filter** in X if $\mathcal{F} \neq \varnothing$, $\varnothing \notin \mathcal{F}$, $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, and $A \subseteq B$ where $A \in \mathcal{F}$ implies $B \in \mathcal{F}$. Moreover, $\mathcal{F}$ is said to be an **ultrafilter** if $\mathcal{F}$ is a maximal filter (i.e. it is not properly contained in any other filter). This is equivalent to the condition that for any $A \subseteq X$ we have $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.

#2 Filtering out the Finite: Let $\mathcal{F} = \{A \subseteq X \mid X - A \text{ is finite}\}$. Show $\mathcal{F}$ is a filter if and only if X is infinite.

Moreover, in the case that $\mathcal{F}$ is a filter (i.e., X is infinite), explain why $\mathcal{F}$ is never an ultrafilter.

Recall that a topological space X is called **completely regular** if for every closed set $A \subseteq X$ and every point $x \in X$ such that $x \notin A$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(a) = 1$ for all $a \in A$ (i.e. closed sets and point can be separated by continuous functions).

#3 A Completely Regular Compact Problem: Let X be a completely regular space. Let $A, B \subseteq X$ be disjoint closed sets and assume A is compact. Prove there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Comment: This falls under our philosophy that compact sets behave like points.

Hint: Since X is completely regular, we can separate each point $a \in A$ from the closed set B with a continuous function f_a. Notice that the inverse image of the open interval $(-1/2, 1/2)$ under the map f_a is an open set containing a. This should give you an open cover of A. Averaging some finite collection of functions should give you your desired function.