Math 4710/5710 # Homework #2 **Due: Wed., Jan. 29$^{\text{th}}$, 2025**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

A set X equipped with a relation "$\leq$" is **totally ordered** if the relation has the following properties:

Reflexive For all $a \in X$, $a \leq a$.

Anti-Symmetric For all $a, b \in X$, if $a \leq b$ and $b \leq a$, then $a = b$.

Transitive For all $a, b, c \in X$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

Comparability For all $a, b \in X$, either $a \leq b$ or $b \leq a$.

Let X be a totally ordered set with order relation $\leq$. Let $A \subseteq X$.

Lower Bound $x \in X$ is a **lower bound** for A if $x \leq a$ for all $a \in A$.

Upper Bound $x \in X$ is an **upper bound** for A if $x \geq a$ for all $a \in A$.

Infimum $x \in X$ is a **greatest lower bound** or **infimum** for A if $x \leq a$ for all $a \in A$ (x is a lower bound) and for any other lower bound $y \in X$ we have $y \leq x$ (x is the largest possible lower bound).

Supremum $x \in X$ is a **least upper bound** or **supremum** for A if $x \geq a$ for all $a \in A$ (x is an upper bound) and for any other upper bound $y \in X$ we have $y \geq x$ (x is the smallest possible lower bound).

Least Upper Bound Property An ordered set has the *least upper bound property* if every non-empty subset that is bounded above has a supremum (in that set).

Greatest Lowest Bound Property An ordered set has the *greatest lower bound property* if every non-empty subset that is bounded below has an infimum (in that set).

#1 Function ?Fun? Let $f : X \to Y$ and $g : Y \to Z$ for some sets X, Y, and Z.

(a) Suppose both f and g are injective. Show $g \circ f$ is too.

(b) Suppose both f and g are surjective. Show $g \circ f$ is too.

(c) Suppose f is bijective and let $B \subseteq Y$. In this case, "$f^{-1}(B)$" could refer to the preimage (=inverse image) of B under f *or* the image of B under f^{-1}. Let's denote the preimage of B (under f) by C and the image of B (under f^{-1}) by D. Show $C = D$ (so there is no real ambiguity when f has an inverse).

#2 Total Order Basics Let X be totally ordered by $\leq$. Let $A \subseteq X$.

(a) Suppose A has an infimum. Show that it is unique (so we can say *the* infimum and denote it by $\inf(A)$).
Note: It is also true (with a very simliar proof) that if A has a supremum, then it must be unique (we denote it by $\sup(A)$).

(b) Give examples of sets with...

i. both supremum and infimum,
ii. no supremum or infimum,
iii. a supremum but no infimum, and
iv. an infimum but no supremum.

(c) [**Grad Problem**] Show that the least upper bound property implies the greatest lower bound property.
Note: The converse is true. This fact has essentially the same proof with all of the $\leq$'s flipped to $\leq$'s.