Math 4710/5710 # Homework #3 **Due: Mon., Feb. 3rd, 2025**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

Recall that if there is an injection from a set A to a set B, we say $|A| \leq |B|$ (i.e., A's cardinality is no more than B's cardinality). If there is a bijection between A and B, we say $|A| = |B|$ (i.e., A and B have the same cardinality). If there is an injection from a set A into $\mathbb{N} = \mathbb{Z}_{\geq 0}$ (i.e., the natural numbers = non-negative integers), then we say A is **countable**. If there is a bijection between A and $\mathbb{N}$, we say A is **countably infinite**. If A is countable but not countably infinite, we say A is **finite**.

Note: We have shorthand notations for countably infinite and continuum (i.e., the size of the real numbers) cardinalities. In particular, $|\mathbb{N}| = \aleph_0$ and $|\mathbb{R}| = 2^{\aleph_0} = \mathfrak{c}$.

#1 A Finite Number of Countable Questions: Determine which sets are countable. Explain your answers.

Undergrads: You may skip 2 of your choice. **Grad students:** Do them all.

(a) The set A of all functions $f : \{0, 1\} \to \mathbb{N}$.

(b) The set B_n of all functions $f : \{1, \ldots, n\} \to \mathbb{N}$ where n is some fixed positive integer.
Note: B_0 would be all functions from the empty set to $\mathbb{N}$. The only such function would be the empty function, so $|B_0| = 1$. In particlar, B_0 is countable but finite...and a weirdo. That's right – you heard me – a weirdo!

(c) The set $C = \displaystyle\bigcup_{n\in\mathbb{N}}^{\infty} B_n$.

(d) The set D of all functions $f : \mathbb{N} \to \mathbb{N}$.

(e) The set E of all functions $f : \mathbb{N} \to \{0, 1\}$.

(f) The set F of all functions $f : \mathbb{N} \to \{0, 1\}$ that are **eventually zero** (i.e., for each f there is some $N > 0$ such that $f(x) = 0$ for all $x > N$).

(g) The set G of all functions $f : \mathbb{N} \to \mathbb{N}$ that are eventually 1.

(h) The set H of all functions $f : \mathbb{N} \to \mathbb{N}$ that are eventually constant.

(i) The set I of all two-element subsets of $\mathbb{N}$.

(j) The set J of all finite subsets of $\mathbb{N}$.

#2 Infinite Calculations: Calculate cardinalities.

Justify your answers using basic cardinal arithmetic and inequalities.

Note: Your answers should either be a non-negative integer (0 or 1 or 2 or ...), $\aleph_0$, $\mathfrak{c} = 2^{\aleph_0}$, or $2^{\mathfrak{c}} = 2^{2^{\aleph_0}}$.

For example: $2^{\aleph_0} \leq \aleph_0^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0^2} = 2^{\aleph_0}$ so that $|\mathbb{Z}^{\mathbb{Z}}| = |\mathbb{Z}|^{|\mathbb{Z}|} = \aleph_0^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$.

(a) $|10^{\mathcal{P}(\mathbb{Z})}|$

(b) $|\mathbb{Z} \times \mathbb{R}^3| + |\mathbb{Q}^{\mathbb{N}}|$

(c) Consider $\mathbb{N}^5$ vs. $5^{\mathbb{N}}$. Do these sets have the same cardinality? Are they countable? (both, either, neither?)