Math 4710/5710 # Homework #4 **Due: Mon., Feb. 17^{th}, 2025**

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#1 Based Question: We have the following lemma (Munkres' Lemma 13.2 on page 80):

Lemma: Let X be a topological space (with topology $\mathcal{T}$). Suppose $\mathcal{C} \subseteq \mathcal{T}$ (i.e., $\mathcal{C}$ is a collection of open sets) such that for each open set U (i.e., $U \in \mathcal{T}$) and $x \in U$, there is some $C \in \mathcal{C}$ such that $x \in C \subseteq U$ (i.e., elements of $\mathcal{T}$ are unions of elements of $\mathcal{C}$). Then $\mathcal{C}$ is a basis that generates $\mathcal{T}$.

(a) Use the above lemma to show $\mathcal{B} = \{(a,b) \mid a < b \text{ where } a,b \in \mathbb{Q}\}$ is a basis for $\mathbb{R}$ (i.e., the reals equipped with the standard topology).

(b) We can also use $\mathcal{B}_{\text{std.}} = \{(a,b) \mid a < b \text{ where } a,b \in \mathbb{R}\}$ as a basis for $\mathbb{R}$. Compare the sizes of $\mathcal{B}$ and $\mathcal{B}_{\text{std.}}$. Show that $\mathbb{R}$ cannot have a basis smaller than $\mathcal{B}$.

(c) Show $\mathcal{C} = \{[a,b) \mid a < b \text{ where } a,b \in \mathbb{Q}\}$ is a basis for the set of real numbers.

(d) **[Grad.]** Show $\mathcal{C}$ does not generate the lower limit topology on $\mathbb{R}$.

#2 Real Weirdness: We discussed the lower limit topology on the reals: $\mathbb{R}_\ell = \mathbb{R}$ has its topology generated by $\mathcal{B}_\ell = \{[a,b) \mid a < b \text{ where } a,b \in \mathbb{R}\}$. Similarly, one can define the upper limit topology on the reals: $\mathbb{R}_u = \mathbb{R}$ where its topology is generated by $\mathcal{B}_u = \{(a,b] \mid a < b \text{ where } a,b \in \mathbb{R}\}$.

(a) In class, we showed $\mathbb{R}_\ell$'s topology is strictly finer than the standard topology. Adapt this argument to show the same is true for $\mathbb{R}_u$ (i.e., $\mathbb{R}_u$'s topology is strictly finer than the standard topology). Then show the lower limit and upper limit topologies are incomparable.

(b) The union of two subbases is still a subbasis. In fact, if $\mathcal{S}$ is a subbasis for X and $\mathcal{S} \subseteq \mathcal{S}' \subseteq \mathcal{P}(X)$, then $\mathcal{S}'$ is still a subbasis (since $\bigcup \mathcal{S}' \supseteq \bigcup \mathcal{S} = X$ so $\bigcup \mathcal{S}' = X$). However, this is not true of bases. Explain why $\mathcal{B}_\ell \cup \mathcal{B}_u$ is not a basis.

(c) **[Grad.]** Let $\mathcal{T}$ be a topology on $\mathbb{R}$ generated by the *subbasis* $\mathcal{B}_\ell \cup \mathcal{B}_u$. This will necessarily be strictly finer than both $\mathbb{R}_\ell$'s and $\mathbb{R}_u$'s topologies. (Why?) Describe $\mathcal{T}$. Better yet, what is $\mathcal{T}$?