Math 4710/5710

Homework #7

Due: Wed., Mar. 19th, 2025

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

Some background: Let us define $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots$ to be the set of sequences of real numbers: $\mathbf{x} = (x_1, x_2, x_3, \dots)$. If we give $\mathbb{R}$ its standard topology, then we can give $\mathbb{R}^\omega$ the corresponding product topology or box topology.

We can also give $\mathbb{R}^\omega$ a *uniform topology* coming from the uniform metric on $\mathbb{R}^\omega$. Here we let:

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_i, y_i) \mid i = 1, 2, \dots\}$$

where $\mathbf{x} = (x_1, x_2, \dots)$, $\mathbf{y} = (y_1, y_2, \dots)$, and $\bar{d}(x_i, y_i) = \min\{|x_i - y_i|, 1\}$. *Note:* If we used $|x_i - y_i|$ instead of $\min\{|x_i - y_i|, 1\}$, our supremum could be ∞ and then $\bar{\rho}$ would not be a metric.

Let $\mathbb{R}^\infty = \{\mathbf{x} \in \mathbb{R}^\omega \mid \mathbf{x} \text{ is eventually zero }\}$. So $\mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^\infty$ implies that there exists some $N \in \mathbb{Z}_{>0}$ such that $x_n = 0$ for all $n \geq N$. So in other words, the coordinates of $\mathbf{x}$ are non-zero only finitely many times.

Note: If we think of $\mathbb{R}^n$ as a subset of $\mathbb{R}^{n+1}$ by identifying $(x_1, \dots, x_n)$ with $(x_1, \dots, x_n, 0)$, then $\mathbb{R}^\infty$ is more-or-less the union of all $\mathbb{R}^n$'s.

#1 One is Enough: We explore these interesting spaces.

(a) Give an example of a subset of $\mathbb{R}^\omega$ that is open in the box topology but not the product topology. You should carefully *prove* your set is *not* open in the product topology.

(b) Assuming we already know $\bar{d}$ is a metric on $\mathbb{R}$, show that $\bar{\rho}$ is a metric on $\mathbb{R}^\omega$.

(c) If $B_\epsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^\omega \mid \bar{\rho}(\mathbf{x}, \mathbf{y}) < \epsilon\}$ denotes the (open) ball centered at $\mathbf{x}$ of radius ϵ, then describe $B_1(\mathbf{0})$ and $B_2(\mathbf{0})$ where $\mathbf{0} = (0, 0, \dots)$.

(d) Determine the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the (i) product, (ii) box, and **[Grad. (iii)]** uniform topologies.