Math 4710/5710 **Homework #8** **Due: Wed., Apr. 2nd, 2025**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

#1 Totally Disconnected: We say a topological space is *totally disconnected* if the only connected subspaces are singleton subsets and the empty set.

(a) Carefully explain why each of the following is totally disconnected:
(i) Any space with the discrete topology (ii) $\mathbb{R}_\ell$ (reals with the lower limit topology) (iii) $\mathbb{Q}$

(b) Let X be given the finite complement topology.
Explain why X is totally disconnected when finite but connected when infinite.

#2 Connecting with Quotients: Let $p : X \to Y$ be a quotient map.
Show X is connected implies Y is connected.

Note: The converse is false. Consider $X = (0,1] \dot{\cup} [2,3)$. Glue the two intervals together so $1 \sim 2$ to get Y. Then $Y = X/\sim$ is homeomorphic to an open interval. Thus the projection map $\pi : X \to Y$ in this case sends a disconnected space to a connected one.

#3 Following the Right Path: Let X and Y be topological spaces.

(a) Suppose X and Y are path connected. Show $X \times Y$ is path connected.
Note: This result is true for arbitrary products.

(b) Let X be path connected and $f : X \to Y$ continuous. Show $f(X)$ is path connected.

(c) Let X_j (for $j \in J$) be path connected spaces such that $\bigcap_{j \in J} X_j \neq \emptyset$. Show $\bigcup_{j \in J} X_j$ is path connected.

(d) **[Grad.]** We had a theorem saying if $A \subseteq B \subseteq \overline{A}$ and A is connected, then B is connected. Is this result true if "connected" is replaced by "path connected"?

#4 Open About Being Connected: [Grad.] Let U be an open and connected subset of $\mathbb{R}^2$.
Show U is path connected.

Suggestion: Consider $u_0 \in U$ and the set $V = \{u \in U \mid u_0 \text{ and } u \text{ are connected by a path in } U\}$.
Show V is clopen.