Math 4710/5710 # Homework #9 **Due: Fri., Apr. 11th, 2025**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

#1 Finitely Many Problems: Let $Y_1, \ldots, Y_n$ be compact subspaces of some topological space X.

(a) Show that $Y_1 \cup \cdots \cup Y_n$ is compact. [A finite union of compact subspaces is still compact.]

(b) Give a counterexample to show that this can fail when we union infinitely many compact subspaces.

#2 Take it to the Limit: Suppose $x_1, x_2, \ldots$ is a sequence in a topological space X such that $x_n \to x$ (i.e., the sequence converges to $x \in X$). Prove $C = \{x\} \cup \{x_1, x_2, \ldots\}$ is compact.

#3 You'll be OK: Recall $K = \{1, 1/2, 1/3, \ldots\}$ and $\mathcal{B}_K = \{(a,b), (a,b) - K \mid a, b \in \mathbb{R} \text{ and } a < b\}$ is a basis for $\mathbb{R}_K$ ($\mathbb{R}$ with the K-topology).

It is helpful to note that if $0 \notin U$, then U is open in $\mathbb{R}_K$ if and only if U is open in $\mathbb{R}$ with the standard topology. Only open neighborhoods of 0 cause problems for us!

(a) Show $I = [0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ is not compact when viewed as a subspace of $\mathbb{R}_K$.
Note: By our note above, since K-open and standard open sets coincide when 0 is not present, any closed interval (e.g., $[1,2]$ or $[-10,-3]$) avoiding 0 would be compact!

(b) Show $\mathbb{R}_K$ is connected.

(c) **[Grad.]** Show $\mathbb{R}_K$ is not path connected.