A_n IS A SIMPLE GROUP (FOR $n \geq 5$)

Definition: G is **simple** if $|G| > 1$ and G has no non-trivial proper normal subgroups.

Theorem: If G is an abelian simple group, then $G \cong \mathbb{Z}_p$ for some prime p.

proof: Let G be an abelian simple group.

Remember that every subgroup of an abelian group is automatically normal (left cosets = right cosets because everything commutes).

Let $x \in G$ with $x \neq 1$ (1 is the identity of G). Then $\langle x \rangle$ is a non-trivial normal subgroup of G. So since G is simple, we must have that $G = \langle x \rangle$ (If G is abelian and simple, G must be cyclic).

First, suppose G is infinite. The only infinite cyclic group (up to isomorphism) is $\mathbb{Z}$. But this is not a cyclic group since $2\mathbb{Z}$ is a non-trivial proper (normal) subgroup. Therefore, G must be finite.

Suppose $|G| = n$. If n is composite, say with proper divisors $k \neq 1$ and $\ell \neq 1$ such that $n = k\ell$, then $x^{n/k}$ is an element of order ℓ. So $\langle x^{n/k} \rangle$ is a non-trivial proper (normal) subgroup of G (of order ℓ). Thus n must be prime (n can't be 1 since simple groups are non-trivial).

Now suppose $|G| = p$ (p prime). Let N be a non-trivial (normal) subgroup of G. Since $|N|$ must divide $|G| = p$ and $|N| \neq 1$ (it's non-trivial). We must have $|N| = p$ and so $N = G$. Thus G has no non-trivial (normal) subgroups. Thus it's simple. $\square$

Lemma: Let N be a normal subgroup of A_n. If N contains a 3-cycle, then $N = A_n$.

proof: Suppose N contains a 3-cycle. We can relabel 1, 2, …, n so this 3-cycle is labeled (123). So without loss of generality assume $(123) \in N$ and so $(123)^2 = (132) \in N$ since N is a subgroup.

If $n = 3$, then $A_3 = \{(1), (123), (123)^2\} \subseteq N$ so $N = A_3$. So assume $n \geq 4$ and pick some $k \geq 4$. $(12)(3k)(132)[(12)(3k)]^{-1} = (12)(3k)(132)(12)(3k) = (12k) \in N$. So $(12k) \in N$ for al $k \geq 3$.

Let a, b, c be distinct numbers between 3 and n. $(1a2) = (12a)(12a) \in N$. $(1ab) = (12b)(12a)(12a) \in N$. $(2ab) = (12b)(12b)(12a) \in N$. $(abc) = (12a)(12a)(12c)(12b)(12b)(12a) \in N$. Thus N contains all 3-cycles.

Finally notice that if a, b, c, d are all distinct, then $(ab)(cd) = (adb)(adc)$ and $(ab)(ac) = (acb)$ and $(ab)(ab) = (1)$ so any permutation written as a product of an even number of transpositions can be written as a product of 3-cycles. Thus A_n is generated by 3-cycles. So if N contains all the 3-cycles, then $N = A_n$. $\square$

Theorem: A_n is simple when $n \geq 5$ and $n = 3$.

proof: Note that A_n only makes sense for $n \geq 2$. A_2 is trivial and A_4 has a proper normal subgroup $H = \{(1), (12)(34), (13)(24), (14)(23)\}$, so A_2 and A_4 are not simple. $A_3 = \langle (123) \rangle \cong \mathbb{Z}_3$ so it's simple (& abelian). Now let $n \geq 5$ and let N be a non-trivial normal subgroup of A_n.

Case 1: N has an element with a cycle of length ≥ 4. Without loss of generality we can relabel 1, 2, …, n so that this cycle is $(123\cdots r)$ for some $r \geq 4$. So there exists some $\sigma = (12\cdots r)\tau \in N$ where $(12\cdots r)$ and τ are disjoint. Consider $(123) \in A_n$ so that $(123)\sigma(123)^{-1} \in N$ since N is normal. Thus $\sigma^{-1}(123)\sigma(123)^{-1} \in N$ since N is a subgroup and thus closed under inverses and the product. $\sigma^{-1}(123)\sigma(123)^{-1} = \tau^{-1}(r\cdots 321)(123)(123\cdots r)\tau(123)^{-1} = \tau^{-1}\tau(r\cdots 321)(2314\cdots r) = (13r) \in N$. Thus N contains a 3-cycle so $N = A_n$.

Case 2: N has an element with a 3-cycle and no cycles of length > 3 (which is covered by case 1). Call this element σ.

First, suppose σ has at least 2 disjoint 3-cycles. Without loss of generality suppose they are (123) and (456) so $\sigma = (123)(456)\tau$ where τ is disjoint from (123) and (456). Consider $(124) \in A_n$. Then $(124)\sigma(124)^{-1} \in N$ and so $\sigma^{-1}(124)\sigma(124)^{-1} \in N$. Thus $\sigma^{-1}(124)\sigma(124)^{-1} = \tau^{-1}(456)^{-1}(123)^{-1}(124)(123)(456)\tau(124)^{-1} = (654)(321)(124)(123)(456)(421) = (14263) \in N$. So N contains a cycle of length > 3. Thus $N = A_n$ by case 1.

Next, suppose σ has 1 cycle of length 3 and then just disjoint transpositions. Without loss of generality suppose this 3-cycle is (123). So $\sigma = (123)\tau \in N$ where τ is the product of disjoint transpositions so that $\tau = \tau^{-1}$. Then $\sigma^2 \in N$ since N is a subgroup. $\sigma^2 = (123)\tau(123)\tau = (123)^2\tau^2 = (123)^2 = (132)$. Thus N contains a 3-cycle so $N = A_n$.

The only possibility left is that σ is just a 3-cycle. But then N contains a 3-cycle so $N = A_n$.

Case 3: N contains an element which is the product of disjoint transpositions. Call it σ. Now since N is a subset of A_n, σ is even. So σ must contain at least 2 disjoint transpositions. Without loss of generality assume these transpositions are (12) and (34). So $\sigma = (12)(34)\tau$ where τ is disjoint from (12) and (34) and $\tau = \tau^{-1}$ since it's the product of disjoint transpositions itself. $(123)\sigma(123)^{-1} \in N$ since N is normal and thus $\sigma^{-1}(123)\sigma(123)^{-1} \in N$ since N is closed under inverses and products. $\sigma^{-1}(123)\sigma(123)^{-1} = \tau^{-1}(34)(12)(123)(12)(34)\tau(132) = (34)(12)(123)(12)(34)(132) = (13)(24) \in N$. So $(135)(13)(24)(135)^{-1} \in N$ and also $(13)(24)(135)(13)(24)(135)^{-1} \in N$. But $(13)(24)(135)(13)(24)(135)^{-1} = (135)$. Thus N contains a 3-cycle so $N = A_n$ [Note: We didn't use the fact that $n \geq 5$ until the very end!] $\square$

Corollary: Let $n \geq 5$. The only normal subgroups of S_n are $\{(1)\}$, A_n, and S_n.

proof: First note that these are in fact normal subgroups of S_n since the trivial subgroup and the whole group are always normal. A_n is the kernel of the sign homomorphism so it's normal [or we could use the fact that A_n is a subgroup of index 2 and index 2 subgroups are always normal].

Let N be a normal subgroup of S_n. Then $N \cap A_n$ is normal in A_n. Thus $N \cap A_n = A_n$ or $N \cap A_n = \{(1)\}$. If $N \cap A_n = A_n$. Then either $N = A_n$ or $|N| > n!/2$ so $|N| = n!$ (there are no divisors of ℓ between $\ell/2$ and ℓ) so $N = S_n$.

Now let's consider the case where $N \cap A_n = \{(1)\}$. Thus $N - \{(1)\}$ is a collection of odd permutations. Let $\sigma, \tau \in N - \{(1)\}$. Then $\sigma\tau \in N$ but $\sigma\tau$ is even since the product of two odd permutations is an even permutation. Thus $\sigma\tau = (1)$. This applies to all non-identity elements of N. So $\sigma\sigma = (1)$ if $\sigma \neq (1)$ in N as well. Thus $\sigma\sigma = (1) = \sigma\tau$ so $\sigma = \tau$. Thus if $N \neq \{(1)\}$, then $N = \{(1), \tau\}$ where $\tau^2 = (1)$. So τ is a product of disjoint transpositions (this must be the case since its order is 2). Also, τ must be odd so it's the product of an *odd* number of disjoint transpositions.

Suppose τ is a single transposition. Without loss of generality assume $\tau = (12)$, then $(13)(12)(13) = (23) \in N$ since N is normal in S_n. But $(12) \neq (13)$ so N has more than 2 elements (contradiction).

Finally consdier the case where τ is the product of more than a single transposition. Without loss of generality assume two the disjoint transpositions are (12) and (34). So $\tau = (12)(34)\sigma$ where σ is disjoint from (12) and (34), then $(13)\tau(13) = (13)(12)(34)\tau(13) = (23)\sigma \in N$. But $\tau = (12)(34)\sigma \neq (23)\sigma$ so N has more than 2 elements (contradiction).

Therefore, N cannot contain a single odd permuation. Thus $N = \{(1)\}$. $\square$