Math 4720 & 5210 # Homework #2 **Due: Fri., Sept. 3^{rd}, 2021**

#1 Establishing Order Redux Recall that $|g|$ denotes the order of g when g is an element of a group. Let G be a group and $a, b \in G$. Prove that $|a| = |bab^{-1}|$ (conjugates have the same order).

[Grad. Students:] Also prove that $|ab| = |ba|$.

Note/Hint: Make sure your proof works for elements of infinite order. Don't assume anything commutes. You may assume this (nearly self-evident) **Lemma:** Let $x, y \in G$ (a group) and suppose that for any $m \in \mathbb{Z}$, $x^m = 1$ iff $y^m = 1$. Then it follows that $|x| = |y|$.

#2 Modding out Happiness Here we practice some modular arithmetic. Recall that $\mathbb{Z}_n^\times = \{k \in \mathbb{Z}_n \mid \gcd(k,n) = 1\}$ is a group under multiplication mod n whereas $\mathbb{Z}_n$ is a (cyclic) group under addition mod n.

(a) How many elements of order 8 are there in $\mathbb{Z}_{23432}$? Find them.

(b) How many elements of order 8 are there in $\mathbb{Z}_{12345}$?

(c) Use the (extended) Euclidean algorithm to show that $56 \in \mathbb{Z}_{12345}^\times$ and actually find 56^{-1}.

For examples of running the (extended) Euclidean algorithm check out answer keys to old Test #1's found with my Math 3110 Exams. In particular, Spring 2021 Test #1 Answer Key Problem 3c gives an example.

You can check your work with https://billcookmath.com/sage/algebra/Euclidean_algorithm.html

#3 Dihedral Fun List all of the cyclic subgroups of D_{10}.

Make a table listing off the possible orders of elements in D_{10} along with how many elements have that order.

Find $Z(D_{10})$. *Note:* Recall that $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$ is the *center* of G.

[Grad. Students:] Find $Z(D_n)$ for all $n \geq 3$. *Hint:* You will want to consider $n = 2k$ and $n = 2k+1$ (even vs. odd) separately.

#4 A Permuted Problem For each of the follow permutations τ, determine if there exists some n-cycle σ such that $\sigma^k = \tau$ for some k.

Example: If $\tau = (13)(24)$, then the answer is "yes". Because given $\sigma = (1234)$ we have $\sigma^2 = (1234)^2 = (13)(24) = \tau$.

(a) $\tau = (12)(34)(56)(78)(9,10)$

(b) $\tau = (12)(345)$

#5 Conjugatin' and Permutatin' Let G be a group and $g \in G$. Recall that the set of all conjugates of g, $\{xgx^{-1} \mid x \in G\}$, is called the *conjugacy class* of g (in G).

(a) List all of the conjugacy classes of S_4.
[Grad. Students:] What are the conjugacy classes of A_4?

(b) When $\sigma \in S_7$, what are the possible orders of σ? Give an example σ for each possible order.

(c) Find the smallest positive integer n such that S_n has an element of order 15. What's the smallest n can be if we wish to have an element of order 16?