Math 4720 & 5210

Homework #3

Due: Mon., Sept. 13^{th}, 2021

#1 An Inverse Problem Let G be a group and let $\varphi : G \to G$ be defined by $\varphi(x) = x^{-1}$ for all $x \in G$. Show that φ is a homomorphism if and only if G is Abelian.

When G is Abelian (so that φ is a homomorphism), is φ an isomorphism? Why or why not?

Note: Since its domain and codomain match, we could call φ an endomorphism if it's a homomorphism. And we could call it automorphism if it's an isomorphism.

#2 Being Productive Let A and B be groups. Recall that $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ becomes a group when we multiply coordinatewise: $(a, b)(c, d) = (ac, bd)$.

(a) Prove that $A \times B \cong B \times A$.

(b) Let $\pi_A : A\times B \to A$ be defined by $\pi_A(a, b) = a$. Show that π_A is an epimorphism (an onto homomorphism). What is its kernel? What does the first isomorphism theorem say?

(c) Show that $A \times B$ is Abelian if and only if *both* A and B are Abelian. Is this still true if we swap out the word "Abelian" with the word "cyclic"?

#3 Cayley Permutes Let G be a group. We write $S(G) = \{f : G \to G \mid f \text{ is a bijection }\}$ for the permutations on G. Let $L_g : G \to G$ be defined by $L_g(x) = gx$ (i.e., left multiplication by g) and $R_g : G \to G$ be defined by $R_g(x) = xg$ (i.e., right multiplication by g).

(a) Show that $L_a \circ R_c = R_c \circ L_a$ for all $a, c \in G$ (just plug in $b \in G$ and compute). In other words, left and right multiplication operators commute. In fact, "left and right multiplication operators commute" is equivalent to which axiom?

(b) Show that $L_g, R_g \in S(G)$ for every $g \in G$.

(c) **[Grad. Problem]** Prove that $\varphi : G \to S(G)$ defined by $\varphi(g) = L_g$ is a monomorphism (i.e., one-to-one homomorphism).

Note: You showed that $L_g \in S(G)$ in the last part, so $S(G)$ makes sense as a codomain.

Observation: You just proved Cayley's Theorem: $G \cong \varphi(G)$ is a subgroup of $S(G)$ (i.e., every group is isomorphic to a subgroup of permutations).

(d) Consider $D_4 = \langle x, y \mid x^4 = 1, y^2 = 1, (xy)^2 = 1\rangle = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$.

Relabel elements as integers: $1 \mapsto 1$, $x \mapsto 2$, $x^2 \mapsto 3$, $\ldots$, $x^3y \mapsto 8$. Under this relabeling L_x becomes the permutation $(1234)(5678)$.

As determined by Cayley's theorem, find a subgroup of S_8 that is isomorphic to D_4.

(e) **[Grad. Problem]** Define $\varphi_g(x) = L_g \circ R_{g^{-1}}(x) = gxg^{-1}$ (i.e., φ_g is conjugation by g). Show φ_g is an automorphism of G.

Note: $\text{Inn}(G) = \{\varphi_g \mid g \in G\}$ is the set of *inner automophisms* and $\text{Aut}(G)$ is the set of automorphisms. It is not hard to show that $\text{Inn}(G)$ is a subgroup of $\text{Aut}(G)$ which in turn is a subgroup of $S(G)$.

(f) **[Grad. Problem]** Let $\varphi : G \to \text{Inn}(G)$ be defined by $\varphi(g) = \varphi_g$ (i.e., g maps to the conjugation by g map). Show that the kernel of φ is the center of G. (What does the isomorphism theorem say?) Also, show that $\text{Inn}(G)$ is a normal subgroup of $\text{Aut}(G)$.

Note: Automorphisms which aren't inner are called "outer automorphisms". In fact, $\text{Aut}(G)/\text{Inn}(G)$ is called the *outer automorphism group* of G.

#4 Permutin' Some More Find a permutation which conjugates $\sigma = (16)(253)(4879)$ to $\tau = (149)(23)(5867)$.

#5 Quotients Write down all of the subgroups and quotients of $\mathbb{Z}_{12}$.

#6 Quotients Again Recall the subgroup lattice of D_4 (as given in class).

(a) List the normal subgroups of D_4. For the non-normal subgroups, show why they fail to be normal by giving left coset which does not match its right coset.

(b) Write down all of the quotients of D_4.