Math 4720 & 5210 # Homework #4 **Due: Mon., Sept. 20th, 2021**

#1 A Classy Problem Write down the class equation for $D_4 = \langle x, y \mid x^4 = 1, y^2 = 1, (xy)^2 = 1 \rangle$.

#2 Normalizing Group Theory We know that G acts on itself via conjugation: $g \bullet x = gxg^{-1}$. This induces an action of G on its powerset $\mathcal{P}(G) = \{A \mid A \subseteq G\}$ where $g \bullet A = gAg^{-1} = \{gag^{-1} \mid a \in A\}$.

(a) Show that the action of G on its powerset is indeed an action (check our two axioms).

Note: Here the stabilizers have a special name: $N_G(A) = \{g \in G \mid gAg^{-1} = A\}$ is the *normalizer* of $A \subseteq G$ in G.

(b) Let $\mathcal{S} = \{H \mid H \text{ is a subgroup of } G\}$. Show that G acts on $\mathcal{S}$ (via conjugation).

Hint: You already know that the group action axioms hold (on all subsets). You essentially just need to verify a kind of closure: show H a subgroup implies gHg^{-1} is a subgroup.

Note: If two subgroup lie in the same orbit, we call them *conjugate subgroups*. That is H and K are conjugate subgroups iff there is some $g \in G$ such that $K = gHg^{-1}$.

(c) In the previous part you showed that G acts on its subgroups via conjugation. What does it mean if the orbit of a subgroup H is $\{H\}$ (a singleton orbit)?

(d) Let H be a subgroup of G. Suppose that K is also a subgroup and that $H \triangleleft K$ (i.e., H is normal in K). Show that $K \subseteq N_G(H)$.

Note: This says that the largest subgroup in which H remains a normal subgroup is the normalizer of H.

(e) Find the normalizers of all of the subgroups of S_3. *Hint:* We always have $H \subseteq N_G(H)$.

(f) Find the normalizers of all of the subgroups of D_4.

(g) **[Grad. Problem]** Find $N_{D_8}(\langle x^4, y \rangle)$.

#3 Permutatin' Sets Let $X = \{1, 2, \ldots, n\}$ and $1 \leq k \leq n$. Let $A \subseteq X$ and $\sigma \in S_n$. Define $\sigma \bullet A = \{\sigma(a) \mid a \in A\}$. For example, $\sigma \bullet \{x, y, z\} = \{\sigma(x), \sigma(y), \sigma(z)\}$.

(a) **[Grad. Problem]** Let $\mathcal{K} = \{A \subseteq X \mid A \text{ has cardinality } k\}$. Show that S_n acts on $\mathcal{K}$ via the operation defined above.

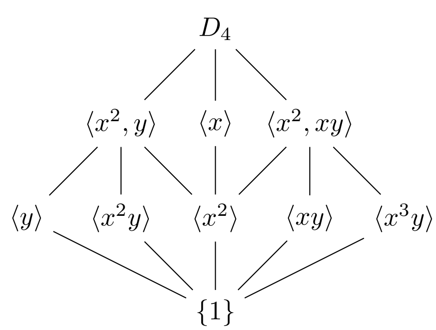
Note: Don't forget to check "closure" as well as the other axioms.

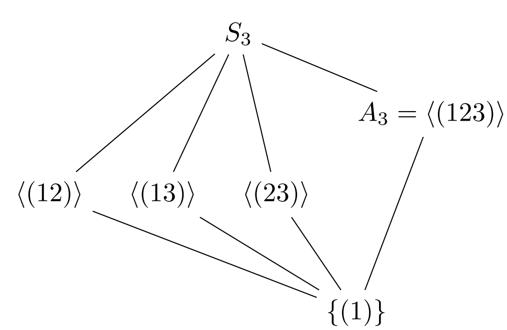
(b) Consider $X = \{1, 2, 3, 4\}$. Describe the action of $\sigma = (12)$ and $\tau = (123)$ on the 2-element subsets of X (there are 6 such subsets).

(c) Find the orbit of $A = \{1, 3\}$ under the action of S_4 on the 2-element subsets of X. Also, find the stabilizer of A.

Note: The orbit will be a *set of sets*.

To help determine normalizers and for future reference, I have included several subgroup lattices. First, we have the subgroup lattices for $D_4 = \langle x, y \mid x^4 = 1, y^2 = 1, (xy)^2 = 1 \rangle = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$ and S_3:





Next, we have the subgroup lattice for $D_8 = \langle x, y \mid x^8 = 1, y^2 = 1, (xy)^2 = 1\rangle = \{1, x, \ldots, x^7, y, xy, \ldots, x^7y\}$: