Math 4720 & 5210 # Homework #5 **Due: Wed., Oct. 6^{th}, 2021**

#1 Units and Zero Divisors Let R be a ring with 1. Recall that $R^\times = \{u \in R \mid u^{-1} \text{ exists }\}$ is the group of units. Also, if $a, b \in R$ such that $a \neq 0$, $b \neq 0$ but $ab = 0$, then we call a a *left zero divisor* and b a *right zero divisor*. A *zero divisor* is either a left or right zero divisor (or both).

(a) Let $u \in R^\times$. Show that $-u \in R^\times$.

(b) Suppose that $r \in R$ is a zero divisor. Show that r is not a unit.
Note: You need to consider two cases. One case where r is a left zero divisor and another where r is a right zero divisor.

(c) Now suppose that R is commutative. Let $\mathcal{L}_r : R \to R$ where $0 \neq r \in R$ and $\mathcal{L}_r(x) = rx$. So $\mathcal{L}_r$ is left (and also because R is commutative, right) multiplication by r. Show that $\mathcal{L}_r$ is one-to-one and onto if and only if r is a unit. Show that $\mathcal{L}_r$ is one-to-one if and only if r is not a zero divisor.

#2 A Centrist Approach Let R be a ring and define $Z(R) = \{r \in R \mid rx = xr \text{ for all } x \in R\}$. This is the center of the ring R.

(a) Show that $Z(R)$ is a subring of R. Moreover, if R is a ring with 1, so is $Z(R)$.

(b) Let D be a division ring (i.e., skew field). Show that $Z(D)$ is a field.

(c) What is the center of the quaternions: $Z(\mathbb{H})$?

#3 Not the Same Let R and S be rings with 1.

(a) Show that if $R \cong S$ then $R^\times \cong S^\times$ (isomorphic rings have isomorphic groups of units).

(b) Show that $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.

#4 An Ideal Problem Let I_i where $i = 1, 2, \ldots$ be ideals of R.

(a) Prove that $\bigcap_{i=1}^{\infty} I_i$ is an ideal of R.

(b) Now assume $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ (this is a *chain of ideals*). Prove that $I = \bigcup_{i=1}^{\infty} I_i$ is an ideal of R.

#5 Sticking to my Principals Let R be a commutative ring with 1 and let $a \in R$. Recall that $(a) = \{ra \mid r \in R\}$ is the principal ideal generated by a.

(a) Prove that (a) is an ideal.

(b) List all of the (distinct) principal ideals in $\mathbb{Z}_{12}$. Show their contents. Example: $(0) = \{0\}$.

(c) Let R be an integral domain. Show that $(a) = (b)$ if and only if a and b are associates (i.e., $a = ub$ for some unit $u \in R^\times$).