Math 4720/5210

Homework #7

Due: Wed., Oct. 27^{th}, 2021

#1 Easy Euclid Let E be a Euclidean domain equipped with norm $\delta : E - \{0\} \to \mathbb{Z}_{\geq 0}$. Suppose $a, b \in E$ are non-zero are non-zero associates. Show that $\delta(a) = \delta(b)$. Is the converse true?

#2 Euclid's Revenge! A quotient of $\mathbb{Q}[x]$.

(a) Find the GCD of $x^3 - 2x^2 + 1$ and $x^2 - x + 3$ in $\mathbb{Q}[x]$ and express it as a linear combination (i.e. run the Extended Euclidean Algorithm).

(b) Let $I = (x^2 - x + 3)$. Is $x^3 - 2x^2 + 1 + I$ zero, a zero divisor, or a unit in $\mathbb{Q}[x]/I$? Prove your result (If zero, why? If a zero divisor, what is a non-zero element that multiplied by gives zero? If a unit, what's its inverse?).

(c) Let $I = (x^3 - 8)$. Is $x^2 - 4 + I$ zero, a zero divisor, or a unit in $\mathbb{Q}[x]/I$? Prove your result (If zero, why? If a zero divisor, what is a non-zero element that multiplied by gives zero? If a unit, what's its inverse?).

#3 A Rational Problem As in the Factorization Handout, compute the inverse of $x^2+x+2+I$ in $\mathbb{Q}[x]/I$ where $I = (x^3-3)$.

Then use this result to rationalize the fraction $\dfrac{1}{2 + 3^{1/3} + 3^{2/3}}$ (i.e. write this fraction as $a + b \cdot 3^{1/3} + c \cdot 3^{2/3}$ for some $a, b, c \in \mathbb{Q}$).

#4 Prime, maximal, both, or neither? Identify the following ideals as prime, maximal, both, or neither.

(a) $(x^2 - 5)$ in $\mathbb{Q}[x]$

(b) $(x^2 - 5)$ in $\mathbb{R}[x]$

(c) $(x^2 + 1)$ in $\mathbb{Q}[x]$

(d) $(x^2 + 1)$ in $\mathbb{Z}[x]$