# Homework #8

**Due: Wed., Nov. 3rd, 2021**

**#1 Irreducible Problems** Let $\mathbb{F}$ be a field.

(a) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{F}[x]$ be irreducible (where $a_n \neq 0$ and $n > 0$).
Prove that $g(x) = a_0 x^n + \cdots + a_{n-1} x + a_n$ (reverse the order of the coefficients) is irreducible in $\mathbb{F}[x]$ as well.
*A Hint and a note:* $x^n f(1/x) = g(x)$ is called the *reciprocal polynomial* of $f(x)$.

(b) Use part (a) and Eisenstein's criterion to show $h(x) = 12x^5 - 24x^4 + 6x^2 + 18x - 1$ is irreducible in $\mathbb{Q}[x]$.

(c) Show $\ell(x) = 5x^3 + 9x^2 - 2x + 2$ is irreducible in $\mathbb{Q}[x]$ by reducing modulo $p$ for some prime $p$.

(d) We could also show that $\ell(x)$ is irreducible using the rational root theorem. Sketch out such a proof.
[You don't actually have go through the trouble of plugging numbers into $\ell(x)$.]

**#2 Bob the Builder** Construct the field of order 8, $\mathbb{F}_8$. In particular, write down its addition and multiplication tables. Also, we know that $\mathbb{F}_8^\times$ is cyclic. Find a generator.

*Note:* Please adopt notation like that used in our finite fields handout where I construct $\mathbb{F}_4$.

**#3 Irreducibly Fun!** First, a quick note: When looking at factorizations, we can treat associates as essentially equal. So when factoring polynomials in $\mathbb{F}[x]$ (where $\mathbb{F}$ is a field), we can focus on monic polynomials. Why? Given an non-zero polynomial, we can multiply it by the multiplicative inverse of its leading coefficient. This gives us a monic associate (i.e., all non-zero polynomials have a *unique* monic associate).

(a) Suppose $f(x) \in \mathbb{F}[x]$ and $\deg(f(x)) = 4$ or $5$. Explain why if $f(x)$ has no roots and no monic irreducible quadratic factors, then it must be irreducible.

(b) List all of the monic irreducible polynomial of degree at most 2 in $\mathbb{Z}_3[x]$.

(c) Show that $f(x) = x^5 + x^4 + 2x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Z}_3[x]$.

(d) Identify the ring $\mathbb{Z}_3[x]/(x^5 + x^4 + 2x^3 + x^2 + x + 1)$. (i.e., What is it?) *Note:* Since we have part (c), no serious computations are needed to answer this question.