Math 4720/5210 # Homework #9 **Due: Mon., Nov. 22^{nd}, 2021**

#1 All Torsion-ed up Inside Let R be a ring with 1 and M an R-module. We call $m \in M$ a *torsion element* if $r \cdot m = 0$ for some $0 \neq r \in R$ (i.e., m is annihilated by some nonzero element of our ring).

Define: $\text{Tor}_R(M) = \{m \in M \mid \exists\, 0 \neq r \in R \text{ such that } r \cdot m = 0\}$.

We showed that $\text{Tor}_R(M)$ is a submodule of M if R is an integral domain. If R is not an integral domain, we can still define this *set* but it may not be a submodule.

(a) Show that if R has zero divisors and $M \neq \{0\}$, then $\text{Tor}(M) \neq \{0\}$.

(b) Let $\varphi : M \to N$ be a R-module homomorphism. Show that $\varphi(\text{Tor}_R(M)) \subseteq \text{Tor}_R(N)$.

(c) **[Grad. Students]** Give an example of a ring R and R-module M where $\text{Tor}(M)$ is not a submodule of M.
Hint: This can be done with some $\mathbb{Z}_n$ as a $\mathbb{Z}_n$-module.

#2 Am I Extendable? Let G be a **finite** abelian group. Then G is a $\mathbb{Z}$-module. Can the action of $\mathbb{Z}$ be extended to an action of $\mathbb{Q}$ making G into a $\mathbb{Q}$-module? Why or why not?

#3 Homies [Grad. Students] Let R be a commutative ring with 1 and let M be an R-module.

Show that $\text{Hom}_R(R, M) \cong M$ as R-modules.

Hint: A homomorphism from a free module to some other module is determined by its action on a basis.

#4 Irreducible Fun! Let R be a ring with 1 and M an R-module and N a submodule of M. Then N is called a *cyclic* R-module if there exists some $n \in N$ such that $N = \langle n \rangle = \text{span}_R\{n\} = R \cdot n = \{r \cdot n \mid R\}$. An R-module M is called *irreducible* if $M \neq \{0\}$ and the only submodules of M are $\{0\}$ and M itself.

(a) Let $x \in M$. Show that $\langle x \rangle$ is a submodule of M.

(b) Show that M is irreducible if and only if $M \neq \{0\}$ and $M = \langle x \rangle$ for all $0 \neq x \in M$ (i.e., any non-zero element can serve as a generator).

(c) **[Grad. Students]** Let R be commutative with 1. Show that M is irreducible if and only if $R/I \cong M$ (as R-modules) for some maximal ideal $I \triangleleft R$.
Suggestion: First, consider why there must be some homomorphism $0 \neq \varphi : R \to M$ and how this gives $R/I \cong M$ for some ideal. Then consider why this only happens when I is maximal.

(d) Determine all of the irreducible $\mathbb{Z}$-modules.
Hint: Consider part (b)'s result.

(e) Let M_1 and M_2 be irreducible R-modules and $\varphi : M_1 \to M_2$ a R-module homomorphism. Show that either $\varphi = 0$ or φ is an isomorphism.
Note: This result is called *Schur's Lemma*. As a consequence, if $M = M_1 = M_2$ is irreducible, we have that every $\varphi \in \text{End}_R(M) = \text{Hom}_R(M, M)$ is either the zero map or an isomorphism. Now, $\text{End}_R(M)$ is a ring under addition of homomorphisms and function composition, it is not the zero ring since the identity map (a homomorphism) is not the zero map (since $M \neq \{0\}$ which is part of the definition of irreducibility), and we have that every non-zero homomorphism is invertible (i.e., a unit in $\text{End}(M)$). Thus $\text{End}_R(M)$ is a *division ring* when M is an irreducible R-module.