**Math 4720 & 5210** # Homework #1 **Due: Fri., Jan. $25^{\text{th}}$, 2019**

**#1 Establishing Order** Find the order and inverse of each of the following elements:

(a) Find the order and inverse of $(4,7)$ in $\mathbb{Z}_5 \times \mathbb{Z}_{12}$. Also, what is the order of the group $\mathbb{Z}_5 \times \mathbb{Z}_{12}$?

*Note:* $\mathbb{Z}_n$ is a group under addition modulo $n$. You should be adding.

(b) Find the order and inverse of $(4,7)$ in $(\mathbb{Z}_5)^\times \times (\mathbb{Z}_{12})^\times$. Also, what is the order of the group $(\mathbb{Z}_5)^\times \times (\mathbb{Z}_{12})^\times$?

*Note:* $(\mathbb{Z}_n)^\times = U(\mathbb{Z}_n) = \{x \in \mathbb{Z}_n \mid \gcd(x,n) = 1\}$ (Gallian calls this $U(n)$) is the group of units of $\mathbb{Z}_n$. This is a group under multiplication modulo $n$. For example: $(\mathbb{Z}_{12})^\times = \{1,5,7,11\}$. You should be multiplying.

(c) Find the order and inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$ in $(\mathbb{Z}_{10})^{2\times 2}$. Also, what is the order of the group $(\mathbb{Z}_{10})^{2\times 2}$?

*Note:* $(\mathbb{Z}_m)^{n\times n}$ is a group under (matrix) addition with entries added modulo $m$. You should be adding.

(d) Find the order and inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$ in $\mathrm{GL}_2(\mathbb{Z}_{10})$.

[~~Grad Students~~ **Bonus Problem:**] Also, what is the order of the group $\mathrm{GL}_2(\mathbb{Z}_{10})$?[1]

> *Note:* I didn't pay attention to the (obvious) fact that $\mathbb{Z}_{10}$ is not a field, so finding the order of $\mathrm{GL}_2(\mathbb{Z}_{10})$ is not that straight forward. However, it isn't that bad once we consider: $k$ and $\ell$ are relatively prime implies $\mathbb{Z}_{k\ell} \cong \mathbb{Z}_k \times \mathbb{Z}_\ell$ (as rings). Next, as rings we get $(\mathbb{Z}_{k\ell})^{2\times 2} \cong (\mathbb{Z}_k)^{2\times 2} \times (\mathbb{Z}_\ell)^{2\times 2}$ so that it follows for their group of units: $\mathrm{GL}_2(\mathbb{Z}_{k\ell}) \cong \mathrm{GL}_2(\mathbb{Z}_k) \times \mathrm{GL}_2(\mathbb{Z}_\ell)$. For the problem at hand, one considers $k = 2$ and $\ell = 5$. Finding the order of $\mathrm{GL}_2$ over a field is relatively straight forward.

*Note:* $\mathrm{GL}_2(\mathbb{Z}_{10}) = ((\mathbb{Z}_m)^{n\times n})^\times = \{A \in (\mathbb{Z}_m)^{n\times n} \mid \det(A) \in (\mathbb{Z}_m)^\times\}$ is a group under (matrix) multiplication with entries computed modulo $m$. You should be multiplying.

*Suggestion:* Even working mod 10, computing the order of this matrix can be tedious (doable but tedious). You might want to use some technology. For example, the code below loads Maple's linear algebra package, defines our matrix, and then computes its square.

```
[> with(LinearAlgebra):
[> A := <<1,3>|<2,9>>;
[> A^2;
```

**#2 Abel Approves** Use the classification of finitely generated Abelian groups.

(a) List all possible Abelian groups of order $4500 = 2^2 \cdot 3^2 \cdot 5^3$ (up to isomorphism). Write each group using both invariant factors and elementary divisors.

(b) Let $G$ be an Abelian group of order $81 = 3^4$. (i) $G$ is not cyclic but does have an element of order 9. What might $G$ be isomorphic to? (ii) What if $G$ does not have any elements of order 9?

**#3 Progressive Abstract Algebra – Intersectionality** Let $H$ and $K$ be subgroups of a group $G$. Prove that $H \cap K$ is a subgroup of $G$.

**[Grad Students:]** Instead of two subgroups, do this for an arbitrary collection of subgroups. For each $i$ in some index set $I$, let $H_i$ be a subgroup of $G$. Prove that $\displaystyle\bigcap_{i\in I} H_i$ is a subgroup of $G$.

**#4 Lemprob Wardsback** Let $\varphi : G_1 \to G_2$ be a homomorphism between two groups. Let $K$ be a subgroup of $G_2$. Prove that $\varphi^{-1}(K)$ is a subgroup of $G_1$.

*Note:* $\varphi^{-1}(K) = \{x \in G_1 \mid \varphi(x) \in K\}$ is the *inverse image* (or *preimage*) of $K$. $\quad x \in \varphi^{-1}(K)$ iff $\varphi(x) \in K$

**Warning:** Given some $x \in G_2$, "$\varphi^{-1}(x)$" is nonsense. The mapping $\varphi$ may not be invertible, so $\varphi^{-1}$ may not be defined *on elements*. On the other hand, $\varphi^{-1}(S)$ *always* makes sense for any subset, $S$, of the codomain.

**#5 You're free cheesybread! You're free! [Grad Students:]** Use the universal property to show that each $x \in X$ is an element of infinite order in the free group $F(X)$. In particular, if $X$ is a non-empty set of generators, we have that $F(X)$ is an infinite group.

Next, suppose $X = \{a,b,c\}$. Show that $ab^{-3}a^2c$ has infinite order in $F(X)$.

*Note:* Actually, all non-identity elements are of infinite order in $F(X)$.

[1] Juncheol Han's paper "The general linear group over a ring", Bulletin of the Korean Mathematical Society, Vol. 43, No. 3 (2006).