**Math 4720 & 5210** **Homework #4** **Due: Mon., Feb. $18^{\text{th}}$, 2019**

**#1 A Classy Problem** Write down the class equation for $S_4$. [**Grad. Problem**] Write $A_4$'s class equation.

**#2 Normalizing Group Theory** We know that $G$ acts on itself via conjugation: $g \bullet x = gxg^{-1}$. This induces an action of $G$ on its powerset $\mathcal{P}(G) = \{A \mid A \subseteq G\}$ where $g \bullet A = gAg^{-1} = \{gag^{-1} \mid a \in A\}$.

(a) Show that the action of $G$ on its powerset is indeed an action (check our two axioms).
*Note:* Here the stabilizers have a special name: $N_G(A) = \{g \in G \mid gAg^{-1} = A\}$ is the *normalizer* of $A \subseteq G$ in $G$.

(b) Let $\mathcal{S} = \{H \mid H \text{ is a subgroup of } G\}$. Show that $G$ acts on $\mathcal{S}$ (via conjugation).
*Hint:* You already know that the group action axioms hold (on all subsets). You essentially just need to verify a kind of closure: show $H$ a subgroup implies $gHg^{-1}$ is a subgroup.
*Note:* If two subgroup lie in the same orbit, we call them *conjugate subgroups*. That is $H$ and $K$ are conjugate subgroups iff there is some $g \in G$ such that $K = gHg^{-1}$.

(c) In the previous part you showed that $G$ acts on its subgroups via conjugation. What does it mean if the orbit of a subgroup $H$ is $\{H\}$ (a singleton orbit)?

(d) Let $H$ be a subgroup of $G$. Suppose that $K$ is also a subgroup and that $H \triangleleft K$ (i.e., $H$ is normal in $K$). Show that $K \subseteq N_G(H)$.
*Note:* This says that the largest subgroup in which $H$ remains a normal subgroup is the normalizer of $H$.

(e) Find the normalizers of all of the subgroups of $S_3$. *Hint:* We always have $H \subseteq N_G(H)$.

(f) Find the normalizers of all of the subgroups of $D_4$.

(g) [**Grad. Problem**] Find $N_{D_8}(\langle x^4, y\rangle)$.

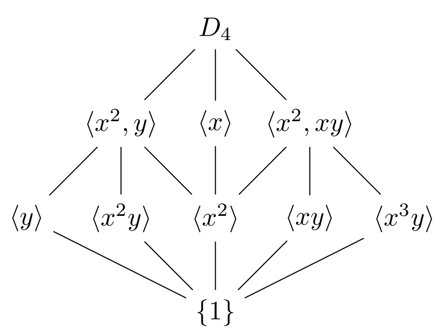
**#3 Permutatin' Sets** Let $X = \{1, 2, \ldots, n\}$ and $1 \leq k \leq n$. Let $A \subseteq X$ and $\sigma \in S_n$. Define $\sigma \bullet A = \{\sigma(a) \mid a \in A\}$. For example, $\sigma \bullet \{x, y, z\} = \{\sigma(x), \sigma(y), \sigma(z)\}$.

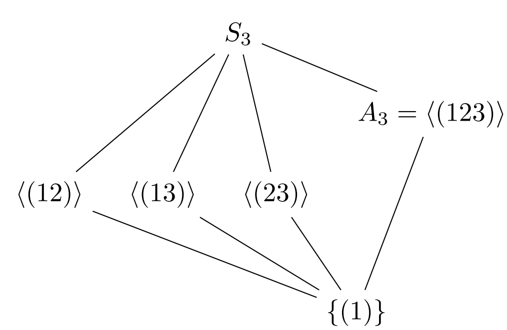
(a) [**Grad. Problem**] Let $\mathcal{K} = \{A \subseteq X \mid A \text{ has cardinality } k\}$. Show that $S_n$ acts on $\mathcal{K}$ via the operation defined above.
*Note:* Don't forget to check "closure" as well as the other axioms.

(b) Consider $X = \{1, 2, 3, 4\}$. Describe the action of $\sigma = (12)$ and $\tau = (123)$ on the 2-element subsets of $X$ (there are 6 such subset).

(c) Find the orbit of $A = \{1, 3\}$ under the action of $S_4$ on the 2-element subsets of $X$. Also, find the stabilizer of $A$.
*Note:* The orbit will be a *set of sets.*

To help determine normalizers and for future reference, I have included several subgroup lattices. First, we have the subgroup lattices for $D_4 = \langle x, y \mid x^4 = 1, y^2 = 1, (xy)^2 = 1\rangle = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$ and $S_3$:



Next, we have the subgroup lattice for $D_8 = \langle x, y \mid x^8 = 1, y^2 = 1, (xy)^2 = 1 \rangle = \{1, x, \ldots, x^7, y, xy, \ldots, x^7 y\}$: