Math 4720 & 5210 **Homework #6** **Due: Mon., Mar. 25th, 2019**

#1 Prime and Maximal As a quick reminder, in $\mathbb{Z}$ and in $\mathbb{Z}_n$, we know that subgroup = normal subgroup = cyclic subgroup = subring = ideal = principal ideal.

(a) Find all the ideals of $\mathbb{Z}_{24}$ and draw the corresponding lattice. Which ideals are prime? Which are maximal?

(b) Determine which ideals in $\mathbb{Z}$ are prime and which are maximal. [Prove your assertions.]
Note: Don't forget to consider the trivial ideal: $\{0\}$.

#2 Another Ideal Problem Let R be a ring and let I and J be ideals of R.

(a) Show that $I+J = \{i+j \mid i \in I \text{ and } j \in J\}$, $I \cap J$, and $IJ = \{i_1 j_1 + \cdots + i_m j_m \mid m \geq 0; i_k \in I \text{ and } j_k \in J\}$ are ideals of R.

(b) Let $R = \mathbb{Z}$, $I = (9)$, and $J = (12)$. What are $I+J$, $I \cap J$, and IJ?

(c) Let R be a **commutative** ring with 1. We say I and J are coprime (or relative prime or comaximal) if $I + J = R$. Suppose that $I + J = R$ (i.e., I and J are relatively prime). Show that $I \cap J = IJ$ and $R/IJ = R/I \cap J \cong R/I \times R/J$.
Note: This is called Chinese Remaindering.
Hints: $IJ \subseteq I \cap J$ is always true. The reverse containment requires the "coprime" assumption. For the next statement, use the first isomorphism theorem. Also, there exists $i \in I$ and $j \in J$ such that $i + j = 1$ (Why?). This decomposition of 1 will help in proving both statements. Finally, for the isomorphism part, consider $aj + bi$. How is this related to a mod I? b mod J?

#3 Idealistic Divisibility Let R be an integral domain. Recall that a divides b iff b is a multiple of a iff there is some $k \in R$ such that $ak = b$ iff $b \in (a)$ iff $(b) \subseteq (a)$.

(a) Let $a, b \in R$. We say $d \in R$ is a *greatest common divisor* (GCD) of a and b iff d is a common divisor of a and b (i.e., d divides a and d divides b) and also given any other common divisor c (i.e., c divides a and c divides b) we have that c divides d.
Suppose that $(a) + (b) = (a, b) = \{ax + by \mid x, y \in R\}$ is principal, say $(a, b) = (d)$. Show that d is a GCD of a and b.

(b) **[Grad. Students]** Give a similar definition for a *least common multiple* (LCM) of a and b. Show that if $(a) \cap (b) = (\ell)$, then ℓ is an LCM of a and b.

#4 Fractionally Important [Grad. Students] Let R be a principal ideal domain (PID) and let S be a multiplicative subset of R (i.e., $a, b \in S$ implies $ab \in S$) and also assume that $0 \notin S$. Show that RS^{-1} is also a PID. [Recall that $RS^{-1} = \{r/s \mid; r \in R \text{ and } s \in S\}$ is the ring of fractions with numerators in R and denominators in S.]

Hint: Let $\mathcal{I}$ be an ideal of RS^{-1}. Consider $I = \{a \in R \mid \text{there exists some } s \in S \text{ such that } a/s \in \mathcal{I}\}$ (i.e. the set of numerators). Show I is an ideal of R and $IS^{-1} = \{a/s \mid a \in I \text{ and } s \in S\} = \mathcal{I}$.