**#1 All Torsion-ed up Inside** Let $R$ be a ring with 1 and $M$ an $R$-module. We call $m \in M$ a *torsion element* if $r \cdot m = 0$ for some $0 \neq r \in R$ (i.e., $m$ is annihilated by some nonzero element of our ring).

Define: $\text{Tor}_R(M) = \{m \in R \mid \exists\, 0 \neq r \in R \text{ such that } r \cdot m = 0\}$.

(a) Compute $\text{Tor}_{\mathbb{Z}}(\mathbb{Z}_3)$ (considering $\mathbb{Z}_3$ as a $\mathbb{Z}$-module) and $\text{Tor}_{\mathbb{Z}_3}(\mathbb{Z}_3)$ (considering $\mathbb{Z}_3$ as a $\mathbb{Z}_3$-module).

(b) Show that if $R$ is an integral domain, then $\text{Tor}(M)$ is a submodule of $M$.

(c) Show that if $R$ has zero divisors and $M \neq \{0\}$, then $\text{Tor}(M) \neq \{0\}$.

(d) Let $\varphi : M \to N$ be a $R$-module homomorphism. Show that $\varphi(\text{Tor}_R(M)) \subseteq \text{Tor}_R(N)$.

(e) **[Grad. Students]** Give an example of a ring $R$ and $R$-module $M$ where $\text{Tor}(M)$ is not a submodule of $M$.
*Hint:* This can be done with some $\mathbb{Z}_n$ as a $\mathbb{Z}_n$-module.

**#2 Am I Extendable?** Let $G$ be a **finite** abelian group. Then $G$ is a $\mathbb{Z}$-module. Can the action of $\mathbb{Z}$ be extended to an action of $\mathbb{Q}$ making $G$ into a $\mathbb{Q}$-module? Why or why not?

**#3 Homies [Grad. Students]** Let $R$ be a commutative ring with 1 and let $M$ be an $R$-module.

Show that $\text{Hom}_R(R, M) \cong M$ as $R$-modules.

*Hint:* A homomorphism from a free module to some other module is determined by its action on a basis.

**#4 Irreducible Fun!** Let $R$ be a ring with 1 and $M$ an $R$-module and $N$ a submodule of $M$. Then $N$ is called a *cyclic* $R$-module if there exists some $n \in N$ such that $N = \langle n \rangle = R \cdot n = \{r \cdot n \mid R\}$. An $R$-module $M$ is called *irreducible* if $M \neq \{0\}$ and the only submodules of $M$ are $\{0\}$ and $M$ itself.

(a) Let $x \in M$. Show that $\langle x \rangle$ is a submodule of $M$.

(b) Show that $M$ is irreducible if and only if $M \neq \{0\}$ and $M$ is cyclic such that $M = \langle x \rangle$ for all $0 \neq x \in M$.

(c) **[Grad. Students]** Let $R$ be commutative with 1. Show that $M$ is irreducible if and only if $R/I \cong M$ (as $R$-modules) for some maximal ideal $I \lhd R$.
*Suggestion:* First, consider why there must be some $0 \neq \varphi : R \to M$ and how this gives $R/I \cong M$ for some ideal. Then consider why this only happens when $I$ is maximal.

(d) Determine all of the irreducible $\mathbb{Z}$-modules.
*Hint:* Consider part (b)'s result.

(e) Let $M_1$ and $M_2$ be irreducible $R$-modules and $\varphi : M_1 \to M_2$ a $R$-module homomorphism. Show that either $\varphi = 0$ or $\varphi$ is an isomorphism.
*Note:* This result is called *Schur's Lemma*. As a consequence, if $M = M_1 = M_2$ is irreducible, we have that every $\varphi \in \text{End}_R(M) = \text{Hom}_R(M, M)$ is either the zero map or an isomorphism. Now, $\text{End}_R(M)$ is a ring under addition of homomorphisms and function composition, it is not the zero ring since the identity map (a homomorphism) is not the zero map (since $M \neq \{0\}$ which is part of the definition of irreducibility), and we have that every non-zero homomorphism is invertible (i.e., a unit in $\text{End}(M)$). Thus $\text{End}_R(M)$ is a *division ring* when $M$ is an irreducible $R$-module.