Math 4720/5210 # Homework #1 **Due: Mon., Jan. 29th, 2024**

#1 An Intersectional Problem: Intersections play nicely with algebraic things. Unions do not.

Recall:

- $x \in \bigcap_{\alpha \in I} S_\alpha \iff$ for all $\alpha \in I$ we have $x \in S_\alpha$
- $x \in \bigcup_{\alpha \in I} S_\alpha \iff$ there exists some $\alpha \in I$ such that $x \in S_\alpha$

(a) Let R be a ring and suppose for some index set I, we have S_α is a subring of a ring R for all $\alpha \in I$. Show $\bigcap_{\alpha \in I} S_\alpha$ is a subring of R.

(b) Let $\mathbb{F}$ be a field and suppose for some index set I, we have $\mathbb{E}_\alpha$ is a subfield of $\mathbb{F}$ for all $\alpha \in I$. $\bigcap_{\alpha \in I} \mathbb{E}_\alpha$ is a subfield of $\mathbb{F}$.

(c) **[Grad.]** Let S and T be subrings of some ring R. Show that $S \cup T$ is a subring if and only if either $S \subseteq T$ or $T \subseteq S$ (i.e., $S \cup T$ is subring only when it equals either S or T).

#2 Calculus! Well, not really: Let R be a (commutative) ring (with 1) and let $f(x) = r_0 + r_1 x + \cdots + r_n x^n \in R[x]$. We can define the formal derivative[1] of $f(x)$ as follows: $f'(x) = \dfrac{d}{dx}\Big[f(x)\Big] = r_1 + 2r_2 x + \cdots + n r_n x^{n-1}$.

Compactly: $\dfrac{d}{dx}\left[\sum_{i=0}^{n} r_i x^i\right] = \sum_{i=1}^{n} i r_i x^{i-1}$

Prove the derivative is a (linear) derivation on $R[x]$. In other words, show (for all $f(x), g(x) \in R[x]$ and $c \in R$):

- $[f(x) + g(x)]' = f'(x) + g'(x)$
- $[cf(x)]' = cf'(x)$
- $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

Hint/Note: When $f(x) = r_0 + r_1 x + \cdots + r_n x^n$, it is often helpful to adopt the convention that $r_k = 0$ for any $k > n$ and $r_k = 0$ for any $k < 0$. This sometimes simplifies notation in proofs. Also, proving linearity is quite easy. To prove the product rule, you probably will want to use linearity.

Following my hit/note, you probably won't need this fact but as a reminder:

If $f(x) = \sum_{i=0}^{n} r_i x^i$ and $g(x) = \sum_{j=0}^{m} s_j x^j$, then $f(x)g(x) = \sum_{\ell=0}^{m+n} \left(\sum_{k=0}^{\ell} r_k s_{\ell-k}\right) x^\ell$.

#3 Degrees of Difficulty: Let R be a commutative ring with 1 and for $0 \neq f(x) \in R[x]$ let $\partial(f(x)) = \deg(f(x)) =$ the degree of $f(x)$.

(a) Let R be an integral domain, $f(x), g(x) \in R[x]$, and $f(x), g(x) \neq 0$. Briefly explain why the leading coefficient of $f(x)g(x)$ is the product of the leading coefficients of $f(x)$ and $g(x)$. Then justify why $\partial(f(x)g(x)) = \partial(f(x)) + \partial(g(x))$. Use this to prove: If R is an integral domain, then so is $R[x]$.

(b) Consider $R = \mathbb{Z}_4[x]$. Show that $(2x+1)^2 = 1$. What does this say about the results of part (a)?

(c) **[Grad.]** Show that x can be factored: $x = f(x)g(x)$ in $\mathbb{Z}_4[x]$ in such a way that neither $f(x)$ nor $g(x)$ is constant. Can this happen in $R[x]$ when R is an integral domain?

#4 Polynomial Units: Recall, for a ring R, we let $R^\times = \{r \in R \mid r^{-1} \text{ exists}\}$ (i.e., the group of units).

(a) Let R be an integral domain. Show $(R[x])^\times = R^\times$.
Also, what is $(\mathbb{Z}[x])^\times$? What is $(\mathbb{F}[x])^\times$ when $\mathbb{F}$ is a field?

(b) Give an example of an infinite ring whose only unit is 1.

(c) Give an example of a non-constant polynomial in $\mathbb{Z}_4[x]$ that is a unit.[2]

---

[1] This is a totally formal notion of derivative. There is no concept of "limit" in a general ring R. I will also note that, for example, $2r_2x$ is not 2 times r_2x but instead it is the 2nd additive power of r_2x. In other words, $2r_2x = r_2x + r_2x$. This may not show up in your proof, but it is something we should be aware of.

[2] This shows that part (a) can break when our coefficient ring is not a field.