#1 Congruently Yours: Notice that $f(x) = x - 1$, $g(x) = x^2 + 1$, and $h(x) = x^2 - 4$ are pairwise relatively prime in $\mathbb{Q}[x]$. Let $I = (f(x))$, $J = (g(x))$, and $K = (h(x))$.

(a) Find the set of all solutions $m(x) \in \mathbb{Q}[x]$ such that $m(x) + I = 4 + I$, $m(x) + J = 2x + J$.
I suggest you use my SAGE interact to do some Euclidean algorithm stuff.

(b) **[Grad.]** Now solve $m(x) + I = 4 + I$, $m(x) + J = 2x + J$, and $m(x) + K = 3x + 4 + K$.

How? Use Chinese remaindering to solve $m(x) + I = 1 + I$ and $m(x) + J = 2x + J$. You will get some solution set $\ell(x) + IJ$. Then for the grad part solve $m(x) + IJ = \ell(x) + IJ$ and $m(x) + K = 3x + 4 + K$. The undergrad answer isn't too bad. The grad answer is kind of ugly – use technology!

#2 Moving up a Notch: Consider the field $\mathbb{F} = \mathbb{C}(x)$ (i.e., ratio of polynomials with complex coefficients). This is the field of fractions of $R = \mathbb{C}[x]$.

I want to prove that the polynomial $f(Y) = Y^2 - x \in \mathbb{F}[Y]$ is irreducible. Let's do this in a less than efficient manner:

(a) Explain why it is enough to show $f(Y)$ is irreducible in $R[Y]$.

(b) Explain why x is an irreducible element of R (use a quotient ring argument).

(c) Suppose $f(Y)$ is not irreducible in $R[Y]$. Then $f(Y) = (Y - a)(Y - b)$ for some $a, b \in R$. Why is this impossible?

(d) Now we know that $f(Y)$ is irreducible in $\mathbb{C}(x)[Y]$. Thus $\mathbb{E} = \mathbb{C}(x)[Y]/(f(Y))$ is an extension field of $\mathbb{F} = \mathbb{C}(x)$. Essentially, we have tacked some new function onto the field of (complex coefficient) rational functions. What should we call it (i.e., secretly what is Y)?

#3 Irreducibly Annoying: We can detect irreducibility of low degree polynomials (i.e., degree 2 and 3) by seeing if they have any roots. Once we move up to fourth degree polynomials, things get more complicated – a fourth degree polynomial can be the product of two irreducible quadratics and thus have no roots!

(a) Generate a list of all irreducible quadratic and cubic polynomials in $\mathbb{Z}_2[x]$.

(b) There are 16 quartic (fourth degree) polynomials in $\mathbb{Z}_2[x]$. Find the irreducibles.
Note: By plugging in 0 and 1, you can easily eliminate those with roots. Any other reducibles must be products of two irreducible quadratics – which you already know from part (a).

(c) Find the monic irreducible quadratic polynomials in $\mathbb{Z}_3[x]$.

(d) Give an example of a monic irreducible cubic in $\mathbb{Z}_3[x]$.
"Thank you Dr. Cook for not asking us to tediously find all of these too."

#4 A Field with Nine: Build the field of order 9, $\mathbb{F}_9$ – do this by constructing a quotient of $\mathbb{Z}_3[x]$. Next, let α be the coset represented by x. List all nine (distinct) elements and fill out a table as follows:

element $z =$	0	1	$\cdots$
additive inverse $-z =$	0	2	$\cdots$
additive order $\|z\| =$	1	3	$\cdots$
mutiplicative inverse $z^{-1} =$	DNE	1	$\cdots$
multiplicative order $\|z\| =$	DNE	1	$\cdots$

Note: DNE = does not exist.