Math 4720/5210 # Homework #5 **Due: Fri., Mar. 8th, 2024**

#1 Irreducible Problems: Let $\mathbb{F}$ be a field.

(a) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{F}[x]$ be irreducible and $f(x)$ is not a multiple of x.
Prove that $g(x) = a_0 x^n + \cdots + a_{n-1} x + a_n$ (reverse the order of the coefficients) is irreducible in $\mathbb{F}[x]$ as well.
Note/Hint: $x^n f(1/x) = g(x)$ is called the *reciprocal polynomial* of $f(x)$.
Why not a multiple of x? Notice if $f(x) = ax = ax + 0$, then $g(x) = 0x + a = a$.
But constant polynomials are either zero or a unit and thus not irreducible.

(b) Use part (a) and Eisenstein's criterion to show $h(x) = 12x^5 - 24x^4 + 6x^2 + 18x - 1$ is irreducible in $\mathbb{Q}[x]$.

(c) Show $\ell(x) = 5x^3 + 9x^2 - 2x + 2$ is irreducible in $\mathbb{Q}[x]$ by reducing modulo p for some prime p.

(d) We could also show that $\ell(x)$ is irreducible using the rational root theorem. Sketch out such a proof.
[You don't actually have go through the trouble of plugging numbers into $\ell(x)$.]

#2 Classical Formulas: Find the roots of these polynomials using the cubic / quartic formula.

(a) $f(x) = x^3 - 24x^2 - 24x - 25$

(b) **[Grad.]** $f(x) = x^4 - 15x^2 - 20x - 6$
Note: Calculate your resolvent cubic. Reduce the resolvent so you have something of the form $u^3 + qu + r$. Then you may "cheat" and use the rational root test to discover a nice root of $u^3 + qu + r$. Using the cubic formula there is nasty.

#3 Algebraic Difficulties: Let $\mathbb{E}$ be an extension field of $\mathbb{F}$.

(a) Let $\alpha, \beta \in \mathbb{E}$ be algebraic over $\mathbb{F}$ and $\alpha \neq 0$. Prove that $\alpha + \beta$, $\alpha\beta$, and α^{-1} are also algebraic over $\mathbb{F}$.
[*Hint:* Don't try to find polynomials. Instead, use the degree formula prove $\mathbb{F}(\alpha, \beta)$ is finite dimensional over $\mathbb{F}$.]

(b) Let $\mathbb{K} = \{\alpha \in \mathbb{E} \mid \alpha \text{ is algebraic over } \mathbb{F}\}$. Prove that $\mathbb{K}$ is a subfield of $\mathbb{E}$ containing $\mathbb{F}$.

(c) **[Grad.]** Define $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ is algebraic (over } \mathbb{Q})\}$. This is called the field of *algebraic numbers*.
Briefly explain why $\overline{\mathbb{Q}}$ (over $\mathbb{Q}$) is an algebraic extension. Then prove this is **not** a finite extension.

Notation? The bar over $\mathbb{Q}$ denotes a kind of closure. It turns out that $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. This is the smallest extension of $\mathbb{Q}$ such that every polynomial (with coefficients in that extension) splits. For example, the fundamental theorem of algebra says that $\overline{\mathbb{R}} = \mathbb{C}$.