Math 4720/5210 **Homework #7** **Due: Wed., Apr. 3rd, 2024**

#1 The Good and the Bad: Consider the following extensions of $\mathbb{Q}$: $\mathbb{E}_1 = \mathbb{Q}[\sqrt{9}]$, $\mathbb{E}_2 = \mathbb{Q}[\sqrt{8}]$, and $\mathbb{E}_3 = \mathbb{Q}[\sqrt[3]{7}]$.

(a) Compute and carefully justify these extensions' degrees: $[\mathbb{E}_1 : \mathbb{Q}]$, $[\mathbb{E}_2 : \mathbb{Q}]$, and $[\mathbb{E}_3 : \mathbb{Q}]$.

(b) Explicitly write down the automorphisms (including actual formulas) making up each of the Galois groups: $G_1 = \mathrm{Gal}(\mathbb{E}_1/\mathbb{Q})$, $G_2 = \mathrm{Gal}(\mathbb{E}_2/\mathbb{Q})$, and $G_3 = \mathrm{Gal}(\mathbb{E}_3/\mathbb{Q})$.

Carefully explain why G_1 and G_3 are trivial groups whereas G_2 is not.

(c) Suppose we modify the second extension so we have $\mathbb{E}_{2'} = \mathbb{F}[\sqrt{8}]$ extending $\mathbb{F}$ for some subfield of $\mathbb{C}$. Is it possible that $G_{2'} = \mathrm{Gal}(\mathbb{E}_{2'}/\mathbb{F})$ has a different group structure? Can $G_{2'}$ be a bigger group? A smaller group?

#2 Doing My Job: Let $\mathbb{E}/\mathbb{F}$ be an extension of fields and $G = \mathrm{Gal}(\mathbb{E}/\mathbb{F})$. Let $S \subseteq G$ (a sub*set* not necessarily a subgroup). We define $\mathbb{E}^S = \{x \in \mathbb{E} \mid \sigma(x) = x \text{ for all } \sigma \in S\}$. This is the subfield of $\mathbb{E}$ *fixed* by S.

(a) Show $\mathbb{E}^S$ is indeed a subfield of $\mathbb{E}$.

(b) Suppose $T \subseteq S$. Show $\mathbb{E}^T \supseteq \mathbb{E}^S$.

(c) **[Grad.]** Let H be a subgroup of G and $\sigma \in G$. Show $\sigma H \sigma^{-1}$ is a subgroup of G. Also, show $\mathbb{E}^{\sigma H \sigma^{-1}} = \sigma(\mathbb{E}^H)$.
What can be said when $H \triangleleft G$?