Math 5160 # Homework #3 **Due: Wed., Feb. 12^{th}, 2020**

#1 Sine of the Times Recall the following definitions: $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$, $\sinh(z) = \dfrac{e^{z} - e^{-z}}{2}$, $\cosh(z) = \dfrac{e^{z} + e^{-z}}{2}$.

(a) Use the above definition of sine (in terms of exponentials) to show that $\dfrac{d}{dz}\left[\sin(z)\right] = \cos(z)$.

(b) Using the definitions above and the fact that $e^{x+iy} = e^x\cos(y) + ie^x\sin(y)$, show that sine decomposes into real and imaginary parts as follows: $\sin(x + yi) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

(c) From part (a), we already know that $\sin(z)$ is differentiable. However, show that this is the case using part (b) and the Cauchy-Riemann equations.

#2 By the Book Let $f(z) = \mathrm{Im}(z)$. Show that $f'(0)$ does not exist using the limit definition of the derivative. Then show that in fact $f'(z)$ does not exist for any z using the Cauchy-Riemann equations.

#3 Calculus I Let $f(z) = z^3$.

(a) Show $f'(z) = 3z^2$ using the limit definition of the derivative.

(b) Let $z = x + yi$, expand $f(x + iy) = (x + iy)^3 = u(x, y) + iv(x, y)$ into its real and imaginary parts and show that they satisfy the Cauchy-Riemann equations.

(c) Verify that $f'(z) = u_x + iv_x = v_y - iu_y$.

#4 A Weird Function Let $f(x + iy) = 3x^2y + (-x^3 + 3xy^2)i$.

(a) Show that $f'(2)$ exists (and find its value) using the limit definition of the derivative.

(b) Show that $f'(2i)$ does not exist using the limit definition of the derivative.

(c) Find where $f'(z)$ does (and does not) exist using the Cauchy-Riemann equations. Find a formula for $f'(z)$ when it does exist.

#5 A Tangential Issue Recall that $\arctan(z) = \dfrac{i}{2}\log\left(\dfrac{1 - iz}{1 + iz}\right)$ (as a multivalued function).

Note: Because $\sin^2(z) + \cos^2(z) = 1$ we also have that $\tan^2(z) + 1 = \sec^2(z)$. Thus because $\tan(\arctan(z)) = z$ we have that $\sec^2(\arctan(z)) = \tan^2(\arctan(z)) + 1 = z^2 + 1$.

(a) Use the technique found in the example on page 52 of Gamelin, to derive the formula for the derivative of $\arctan(z)$ (i.e. differentiate $\tan(\arctan(z)) = z$ etc.).

(b) Rederive the formula for the derivative of $\arctan(z)$ using its formula stated above (in terms of the complex logarithm) along with the fact that the derivative of $\log(z)$ is $1/z$.