Math 5160 # Homework #4 **Due: Wed., Feb. 19^{th}, 2020**

#1 A Harmonious Problem

(a) In the last homework we found the real and imaginary parts of $\sin(z)$ (#1 Sine of the Times part (b)). We also showed that the derivative of sine was cosine. Verify directly that the real and imaginary parts of $\sin(z)$ are harmonic functions.

(b) Expand $\cosh(z)$ into its real and imaginary parts and verify that they are harmonic functions.

(c) Show that $u(x,y) = x^3 - 3xy^2 + 2x^2 - 2y^2 + 3y$ is harmonic and then find a harmonic conjugate for $u(x,y)$.

(d) We know that if $v(x,y)$ is a conjugate for $u(x,y)$, then $f(z) = u + iv$ is analytic. Referring to part (c), find a nice formula for f which only involves z.

#2 You Must Conform You should probably use some software to help you generate nice plots. I recommend my Maple worksheet.

(a) Explain where $g(z) = \text{Log}(z)$ is a conformal mapping. Give parametrizations for the circles $x^2 + y^2 = 16$ and $(x-3)^2 + (y-2)^2 = 4$ and a parameterization for the line through $2i$ and 2. Then plot these curves and their images under the map $g(z) = \text{Log}(z)$. Adjust parameter domains to make your plots look nice.

(b) Explain where $g(z) = z^3 + 3z^2 - 9z + 1$ is a conformal mapping. Give parametrizations for the circles $x^2 + y^2 = 9$ and $(x+1)^2 + y^2 = 4$ and a parameterization for the line $y = 2x - 1$. Then plot these curves and their images under the map $g(z) = z^3 + 3z^2 - 9z + 1$. Adjust parameter domains to make your plots look nice.

#3 Möbius Just Sounds Cool Let $f_A(z) = \dfrac{a_{11}z + a_{12}}{a_{21}z + a_{22}}$ where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\det(A) \neq 0$.

(a) Find a formula for the linear fractional transformation which sends $(1, -1, i)$ to $(\infty, 0, 1)$. Show a detailed construction of such a formula by constructing maps to $(0, 1, \infty)$ and then putting them together to form the desired map (as in the constructive proof from the video). Then create a plot showing some lines, circles mapping to lines, circles.

(b) Repeat part (a) for $(i, \infty, 0)$ to $(-1, i, \infty)$.

(c) Show that $f_A(z) = f_B(z)$ if and only if there exists some $s \in \mathbb{C}$, $s \neq 0$, and $B = sA$ (i.e., A and B are off by a nonzero scalar multiple). *Hint:* Consider $f_{AB^{-1}}(z)$.