Homework #4

#1 A Harmonious Problem

- (a) In the last homework we found the real and imaginary parts of $\sin(z)$ (#1 Sine of the Times part (b)). We also showed that the derivative of sine was cosine. Verify directly that the real and imaginary parts of $\sin(z)$ are harmonic functions.
- (b) Expand $\cosh(z)$ into its real and imaginary parts and verify that they are harmonic functions.
- (c) Show that $u(x,y) = x^3 3xy^2 + 2x^2 2y^2 + 3y$ is harmonic and then find a harmonic conjugate for u(x,y).
- (d) We know that if v(x, y) is a conjugate for u(x, y), then f(z) = u + iv is analytic. Referring to part (c), find a nice formula for f which only involves z.
- **#2 You Must Conform** You should probably use some software to help you generate nice plots. I recommend my Maple worksheet.
 - (a) Explain where g(z) = Log(z) is a conformal mapping. Give parametrizations for the circles $x^2 + y^2 = 16$ and $(x-3)^2 + (y-2)^2 = 4$ and a parameterization for the line through 2i and 2. Then plot these curves and their images under the map g(z) = Log(z). Adjust parameter domains to make your plots look nice.
 - (b) Explain where $g(z) = z^3 + 3z^2 9z + 1$ is a conformal mapping. Give parametrizations for the circles $x^2 + y^2 = 9$ and $(x + 1)^2 + y^2 = 4$ and a parameterization for the line y = 2x 1. Then plot these curves and their images under the map $g(z) = z^3 + 3z^2 9z + 1$. Adjust parameter domains to make your plots look nice.

#3 Möbius Just Sounds Cool Let $f_A(z) = \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}$ where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and det $(A) \neq 0$.

- (a) Find a formula for the linear fractional transformation which sends (1, −1, i) to (∞, 0, 1). Show a detailed construction of such a formula by constructing maps to (0, 1, ∞) and then putting them together to form the desired map (as in the constructive proof from the video). Then create a plot showing some lines, circles mapping to lines, circles.
- (b) Repeat part (a) for $(i, \infty, 0)$ to $(-1, i, \infty)$.
- (c) Show that $f_A(z) = f_B(z)$ if and only if there exists some $s \in \mathbb{C}$, $s \neq 0$, and B = sA (i.e., A and B are off by a nonzero scalar multiple). *Hint:* Consider $f_{AB^{-1}}(z)$.