Math 5160 # Homework #6 **Do after Gamelin V.1**

#1 Liouville Slugger Let u be a (real valued) harmonic function on the entire complex plane. Assume that u is bounded above. Show that u is constant.

Note: We discussed harmonic functions and a maximum modulus result earlier. However, I would like you to approach the problem as follows: (1) Express u as part of an analytic function (Why/how can this be done?). (2) Next, exponentiate and use Liouville's Theorem. (3) Make sure you justify your use of any theorems (i.e., Are the hypotheses satisfied?).

#2 Lines Don't Matter Let L be a line in the complex plane. Suppose that $f(z)$ is continuous on a domain D and analytic on $D - L = \{z \in D \mid z \notin L\}$. Show that $f(z)$ is analytic on D.

Hint/Suggestion: Use (don't reprove) the theorem in Gamelin about the special case of $L = \mathbb{R}$.

#3 More–rara Fun State a version of Morera's theorem which uses triangles instead of rectangles. Then use Gamelin's version of Morera's theorem (page 119) to prove your version.

Note: Make your theorem strong like Gamelin's. He doesn't require verification around all rectangles – just certain kinds. What nice kind of triangles can you restrict to?

#4 Fancy Notation Show that $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$.

Note: You may (and should) assume that mixed second partials are continuous to prove the above identity.

Next, let h be a smooth complex-valued function defined on some complex domain D.

(a) Show that h is harmonic on D if and only if $\dfrac{\partial^2 h}{\partial z \partial \bar{z}} = 0$ on D.

(b) Show that h is harmonic on D if and only if $\dfrac{\partial h}{\partial z}$ is analytic on D.

#5 Some Analysis Let z_k be a sequence of complex numbers.

Show that $\sum\limits_{k=0}^{\infty} z_k$ converges absolutely if and only if both $\sum\limits_{k=0}^{\infty} \mathrm{Re}(z_k)$ and $\sum\limits_{k=0}^{\infty} \mathrm{Im}(z_k)$ converge absolutely.