This is an extended example of computing with bases. Let us begin with a subspace W of $\mathbb{R}^5$.

$$W = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} \in \mathbb{R}^5 \;\middle|\; \begin{array}{r} v_1 + 2v_2 - v_3 + v_4 + 5v_5 = 0, \\ v_3 + v_4 - 2v_5 = 0, \\ \\ -v_1 - 2v_2 + 2v_3 - 7v_5 = 0, \end{array} \text{ and } \right\}$$

We might immediately notice that the elements of W satisfy a system of homogeneous equations. In particular, for any $\mathbf{v} \in W$ we have...

$$A\mathbf{v} = \begin{bmatrix} 1 & 2 & -1 & 1 & 5 \\ 0 & 0 & 1 & 1 & -2 \\ -1 & -2 & 2 & 0 & -7 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, $W = N(A)$ (W is the null space of A). So W is indeed a subspace of $\mathbb{R}^5$.

To find a basis for W, we should find the RREF of A and solve the system $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 5 \\ 0 & 0 & 1 & 1 & -2 \\ -1 & -2 & 2 & 0 & -7 \end{bmatrix} \underset{\sim}{\text{G. E.}} \begin{bmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{rcl} x_1 & = & -2r - 2s - 3t \\ x_2 & = & r \\ x_3 & = & -s + 2t \\ x_4 & = & s \\ x_5 & = & t \end{array} \quad \mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} t$$

$$\text{Therefore, } W = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} = \operatorname{Col}(B) \text{ (a column space) where } B = \begin{bmatrix} -2 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover, these 3 vectors form a basis for W, so we now know that $\dim(W) = 3$. Also, notice that $W = N(A) = \operatorname{Col}(B)$, so W can be viewed as both a null space and a column space. While not obvious, it is true that **all** subspaces of $\mathbb{R}^n$ can be viewed as both null spaces and column spaces if we pick the right matrices.

Let's ask the question: "Is $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} \in W$?" There are 2 basic techniques for checking.

- Since W is a nullspace, we can see if "$A\mathbf{w} = \mathbf{0}$" is satisfied. So $\mathbf{w} \notin W$ because...

$$A\mathbf{w} = \begin{bmatrix} 1 & 2 & -1 & 1 & 5 \\ 0 & 0 & 1 & 1 & -2 \\ -1 & -2 & 2 & 0 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -6 \end{bmatrix} \neq \mathbf{0}$$

- Alternatively, we can use the basis that we found above to check is our vector belongs to W. This is a lot more work than our previous technique. However, if we have a subspace described as the "span" of a set of vectors, we are more-or-less forced to check this way. Specifically, we will adjoin the vector in question to the spanning set (which is actually a basis in this case) and then row reduce.

$$[B : \mathbf{w}] = \begin{bmatrix} -2 & -2 & -3 & : & 1 \\ 1 & 0 & 0 & : & 2 \\ 0 & -1 & 2 & : & 3 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \underset{\sim}{\text{G. E.}} \begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

  Since the final column is a pivot column, the final column is **not** contained in the span of the previous columns. Therefore, $\mathbf{w} \notin W$.

  This illustrates that null space membership is easier to check than column space membership.

Next, let's show that $S = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a subset of W where $\mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 8 \\ -4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.
Again let's do this 2 ways. [First the easy way, then the hard way.]

- A quick check shows $A\mathbf{a} = \mathbf{0}$, $A\mathbf{b} = \mathbf{0}$, and $A\mathbf{c} = \mathbf{0}$ so $\mathbf{a}, \mathbf{b}, \mathbf{c} \in W$. So $S \subset W$ (S is contained in W).
- Alternatively, we can adjoin our three vectors to the spanning set from before and row reduce.

$$[B \,:\, \mathbf{a}\,\mathbf{b}\,\mathbf{c}] = \left[\begin{array}{ccc:ccc} -2 & -2 & -3 & 0 & 2 & 8 \\ 1 & 0 & 0 & 1 & -2 & -4 \\ 0 & -1 & 2 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{\text{G. E.}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -2 & -4 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So we put our "known" vectors to the left and the vectors to "test" to the right. Since there are no "new" pivot columns (coming from our test vectors), they must all lie in the span of our known vectors. In particular, $\mathbf{a}$ is just the difference between the first two vectors, $\mathbf{b}$ is -2 times the first vector plus the second vector, and $\mathbf{c}$ is just -4 times the first vector. Thus the new vectors are all elements of W. Hence $S \subset W$.

We already know that S is a subset of W. But is S a linearly independent set? And, can we extend all (or part) of S to a basis for W?

To answer both of these questions (at the same time) we should put the vectors from S into a matrix and adjoin a known basis for W. To guarantee that the vectors in S are placed in the basis we are trying to compute, we should place them on the left (this gives them preferential treatment).

$$\left[\begin{array}{ccc:ccc} 0 & 2 & 8 & -2 & -2 & -3 \\ 1 & -2 & -4 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 2 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{\text{G. E.}} \left[\begin{array}{ccc:ccc} 1 & 0 & 4 & -1 & -2 & 0 \\ 0 & 1 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Notice that the third column is **not** a pivot column. In fact, the linear correspondence says that it is 4 times the first column plus 4 times the second. Thus S is linearly dependent.

$$4\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Next, notice that the fourth and fifth columns are also linear combinations of the first two columns. So they are also redundant. However, the last column is a pivot column, so it is not contained in the span of the first two columns. Adding this vector to the first two will yield a basis for W.

$$\beta = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis for W (which includes as much of S as possible).

Let's note one last thing. If we had listed the vectors from S in a different order, we would end up with a different basis.

$$\left[\begin{array}{ccc:ccc} 8 & 0 & 2 & -2 & -2 & -3 \\ -4 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 2 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{\text{G. E.}} \left[\begin{array}{ccc:ccc} 1 & 0 & 1/4 & -1/4 & -1/4 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies \beta = \left\{ \begin{bmatrix} 8 \\ -4 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This is also a basis for W and it also includes two vectors from S, but not the same vectors as before. This is due to our change in "preferences" (the change in the order in which we listed the vectors).