Gram-Schmidt, QR Decompostion, and Coordinates.

> restart; with(LinearAlgebra):

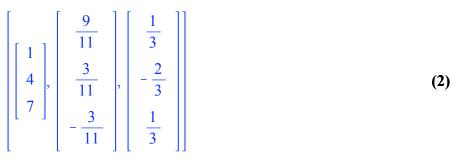
Let's run the Gram-Schmidt orthogonalization process on the columns of the following matrix...

> A := <<1,4,7>|<2,5,8>|<3,6,9>|<3,9,15>|<1,0,1>>;

$$A := \begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 4 & 5 & 6 & 9 & 0 \\ 7 & 8 & 9 & 15 & 1 \end{bmatrix}$$
(1)

The "GramSchmidt" command takes in a list of vectors. It doesn't like to take a matrix as an input, so I'll pull A apart.

```
> GramSchmidt([seq(A[1..RowDimension(A),i],i=1..ColumnDimension(A))])
```



This gives unnormalized vectors.

;

I'll toss in the "normalized" option and convert the list of vectors to a matrix (and call this matrix Q)...

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> Q := Matrix(GramSchmidt([seq(A[1..RowDimension(A),i],i=1..
ColumnDimension(A))], normalized));
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$$Q := \begin{bmatrix} \frac{1}{66} \sqrt{66} & \frac{3}{11} \sqrt{11} & \frac{1}{6} \sqrt{6} \\ \frac{2}{33} \sqrt{66} & \frac{1}{11} \sqrt{11} & -\frac{1}{3} \sqrt{6} \\ \frac{7}{66} \sqrt{66} & -\frac{1}{11} \sqrt{11} & \frac{1}{6} \sqrt{6} \end{bmatrix}$$
(3)

Now let R be the matrix whose (i,j)-entry is the dot product of the i-th column of Q dotted with the j-th row of A. In other words, $R = Q^T A$.

> R := Transpose (0) . A;
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$$R := \begin{bmatrix} \sqrt{66} & \frac{11}{11} & \sqrt{66} & \frac{15}{11} & \sqrt{66} & \frac{24}{11} & \sqrt{66} & \frac{4}{33} & \sqrt{66} \\ 0 & \frac{3}{11} & \sqrt{11} & \frac{6}{11} & \sqrt{11} & \frac{3}{11} & \sqrt{11} & \frac{2}{11} & \sqrt{11} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \sqrt{6} \end{bmatrix}$$
Notice that R is upper-triangular because of the iterative nature of the Gram-Schmidt process.
Specifically, the j-th column of R is built from the j-th column of A which can be expressed as a linear combination of the first up to j-th vectors obtained from Gram-Schmidt.
Finally because Q is orthogonal, its inverse is its transpose. Thus $QR = QQ^T A = A$.
> Q.R;

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 4 & 5 & 6 & 9 & 0 \\ 7 & 8 & 9 & 15 & 1 \end{bmatrix}$$
The following vector belongs to the column space of A:
> v := $\langle 2, 3, 6 \rangle$;

$$v := \begin{cases} 2 \\ 3 \\ 6 \end{bmatrix}$$
(6)
This is revealed by row reduction...
> ReducedRowEchelonForm($\langle A | v \rangle$);

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & \frac{1}{3} \\ 0 & 1 & 2 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$
Notice that v is the 1/3 the first column of A plus 1/3 the second column plus the fifth column of A. This means that relative to the pivot column basis for A, the coordinates of v are...
> colCoordsv := $\langle \langle 1/3, 1/3, 1 \rangle$;

$$colCoordsv := \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$
(8)

Recall that the columns of Q also form a basis for the column space of A. (Gram-Schmidt kicks out a basis - but not just any kind of basis - an orthogonal basis!)

Let's see that v belongs to the column space of A by seeing it expressed in this orthogonal basis....

> ReducedRowEchelonForm(<Q|v>);

Easier than row reduction, we can find coordinates by using dot products. Each coordinate of v (in this orthonormal basis) is nothing more than the dot product of v with a column of Q.

> <DotProduct (v, Q[1..3,2]),
DotProduct (v, Q[1..3,3])>>;
$$\begin{bmatrix} \frac{28}{33} \sqrt{66} \\ \frac{3}{11} \sqrt{11} \\ \frac{1}{3} \sqrt{6} \end{bmatrix}$$
(10)