Let V be a vector space (over $\mathbb{F}$) such that $\dim(V) = n < \infty$. Let $T : V \to V$ be a linear transformation (since we are mapping from V to itself, we could refer to T as a linear endomorphism or a linear operator).

Definition: Let $\mathbf{v} \in V$ such that $\mathbf{v} \neq 0$ and $T(\mathbf{v}) = \lambda\mathbf{v}$. Then $\mathbf{v}$ is an *eigenvector* for T with *eigenvalue* λ. Moreover, we say that $\lambda \in \mathbb{F}$ is an *eigenvalue* for T if T has an eigenvector with eigenvalue λ.

Note: $\mathbf{0}$ is not allowed to be an eigenvector. Otherwise since $T(\mathbf{0}) = \mathbf{0} = \lambda\mathbf{0}$ we would have that every scalar is an eigenvalue of T and $\mathbf{0}$ would have every scalar as its eigenvalue!

Definition: Let $f(t) = \det(tI - T)$. Then $f(t)$ is called the *characteristic polynomial* of T.[1]

Note: λ is an eigenvalue of $T \Leftrightarrow$ there exists an non-zero vector $\mathbf{v}$ such that $T(\mathbf{v}) = \lambda\mathbf{v} \Leftrightarrow$ there exists a non-zero vector $\mathbf{v}$ such that $(\lambda I - T)(\mathbf{v}) = \mathbf{0} \Leftrightarrow \mathrm{Ker}(\lambda I - T) \neq \{\mathbf{0}\} \Leftrightarrow \lambda I - T$ is not 1-1 $\Leftrightarrow \lambda I - T$ is not invertible $\Leftrightarrow \det(\lambda I - T) \neq 0$. We have just proved...

Theorem: λ is an eigenvalue of T if and only if λ is a root of the characteristic polynomial of T (that is $f(\lambda) = \det(\lambda I - T) = 0$).

Facts: Let $f(t)$ be the characteristic polynomial of T. Then $f(t)$ is a polynomial of degree n whose leading coefficient is 1 (i.e. $f(t)$ is a *monic* polynomial). In addition, $f(0) = (-1)^n \det(T)$. Also, the coefficient of t^{n-1} in $f(t)$ is $-\mathrm{tr}(T)$ (minus the trace of T).

Definition: Factor T's characteristic polynomial: $f(t) = (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_k)^{m_k}$ (where $\lambda_i \neq \lambda_j$ for $i \neq j$ and $m_i > 0$). Then the roots of $f(t)$ (i.e. the eigenvalues of T) are $\lambda_1, \ldots, \lambda_k$. We say that the *algebraic multiplicity* of λ_i is m_i (the number of factors $(t-\lambda_i)$ appearing in the characteristic polynomial). Notice that the sum of the algebraic multiplicities is n (the degree of the characteristic polynomial).

Definition: $E_\lambda = \{\mathbf{0}\} \cup \{\mathbf{v} \in V \mid \mathbf{v}$ is an eigenvector of T with eigenvalue $\lambda\}$ That is
$E_\lambda = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \lambda\mathbf{v}\} = \{\mathbf{v} \in V \mid (\lambda I - T)(\mathbf{v}) = \mathbf{0}\} = \mathrm{Ker}(\lambda I - T)$. If $E_\lambda \neq \{\mathbf{0}\}$ (this happens exactly when λ is an eigenvalue), then we call E_λ an *eigenspace* of T. Notice that E_λ is a subspace of V (since it is the kernel of a linear transformation).

Definition: $\dim(E_\lambda) = \dim(\mathrm{Ker}(\lambda I - T)) = \mathrm{nullity}(\lambda I - T)$ is called the *geometric multiplicity* of λ. Notice that if λ is an eigenvalue then the eigenspace cannot be the zero subspace. Thus geometric multiplicities of eigenvalues are always at least 1.

Theorem: Let λ be an eigenvalue of T with algebraic multiplicity m and geometric multiplicity g. Then $1 \leq g \leq m$.

Theorem: Eigenvectors with different eigenvalues are linearly independent. Moreover, if S_i is a linearly independent set of eigenvectors with eigenvalue λ_i and $\lambda_i \neq \lambda_j$ for $i \neq j$, then $S_1 \dot\cup S_2 \dot\cup \cdots \dot\cup S_k$ is a linearly independent set.

Definition: T is *diagonalizable* if there is a basis for V consisting of eigenvectors for T. Notice if β is such a basis, then $[T]_\beta$ is a diagonal matrix!

Corollary: T is diagonalizable (over $\mathbb{F}$) if and only if the eigenvalues of T all belong to $\mathbb{F}$ (i.e. the characteristic polynomial completely factors over $\mathbb{F}$) and the geometric and algebraic multiplicities of each eigenvalue match.

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[1]This is the definition in Hoffman and Kunze. Freidberg, Insel, Spence define $g(t) = \det(T - tI)$ to be the characteristic polynomial. Notice that $f(t) = (-1)^n g(t)$ where $n = \dim(V)$.