Consider the following linear transformations:

$$S : P_2 \to \mathbb{R}^{2\times 2} \qquad \text{defined by} \qquad S(ct^2+bt+a) = \begin{bmatrix} a+b & b \\ c & a+c \end{bmatrix}$$

$$T : \mathbb{R}^{2\times 2} \to \mathbb{R}^2 \qquad \text{defined by} \qquad T\left(\begin{bmatrix} x & y \\ u & v \end{bmatrix}\right) = (x-y, u-v)$$

Notice that if we compose these maps we get $T \circ S : P_2 \to \mathbb{R}^2$ where

$$(T\circ S)(ct^2+bt+a) = T(S(ct^2+bt+a)) = T\left(\begin{bmatrix} a+b & b \\ c & a+c \end{bmatrix}\right) = (a+b-b, c-(a+c)) = (a,-a)$$

Consider the standard bases: $\beta = \{1, t, t^2\}$ for P_2, $\delta = \{e_1 = (1,0), e_2 = (0,1)\}$ for $\mathbb{R}^2$, and

$$\gamma = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ for } \mathbb{R}^{2\times 2}.$$

Let's find coordinate matrices for S, T, and $T \circ S$.

- $S(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1E_{11} + 0E_{12} + 0E_{21} + 1E_{22} \implies [S(1)]_\gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

  $S(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 1E_{11} + 1E_{12} + 0E_{21} + 0E_{22} \implies [S(1)]_\gamma = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

  $S(t^2) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0E_{11} + 0E_{12} + 1E_{21} + 1E_{22} \implies [S(1)]_\gamma = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

  Therefore, $[S]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

- $T(E_{11}) = T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = (1,0) = 1e_1 + 0e_2 \implies [T(E_{11})]_\delta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

  $T(E_{12}) = T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = (-1,0) = -1e_1 + 0e_2 \implies [T(E_{12})]_\delta = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

  $T(E_{21}) = T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = (0,1) = 0e_1 + 1e_2 \implies [T(E_{21})]_\delta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

  $T(E_{22}) = T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = (0,-1) = 0e_1 - 1e_2 \implies [T(E_{22})]_\delta = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

  Therefore, $[T]_\gamma^\delta = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

- To find a coordinate matrix for $T \circ S$ we could do a direct computation like before...or we can use our work from the last two bullets:

  $$[T\circ S]_\beta^\delta = [T]_\gamma^\delta [S]_\beta^\gamma = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

  [Or by direct computation: $(T\circ S)(1) = (1,-1) = 1e_1 - 1e_2$ and $(T\circ S)(t^2) = (T\circ S)(t) = (0,0) = 0e_1 + 0e_2$ which gives us the same matrix (of course).]

Let's find a basis for the Kernel and Range of S, T, and $T \circ S$. We know that if X is a linear transformation with corresponding matrix Y then $\mathrm{N}(Y)$ is a coordinate representation of $\mathrm{Ker}(X)$ and $\mathrm{Col}(Y)$ is a coordinate representation of $\mathrm{Range}(X)$.

- For S we have... $$[S]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

  Thus $\mathrm{N}([S]_\beta^\gamma) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ and so $\mathrm{Ker}(S) = \{0\}$ which means S is 1-1 and $\mathrm{nullity}(S) = 0$.

  Next, we see that every column of the coordinate matrix is a pivot column so that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Col}([S]_\beta^\gamma)$. These coordinate vectors correspond to the following set (which is a basis for $\mathrm{Range}(S)$): $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ which is a basis for $\mathrm{Range}(S)$. Thus $\mathrm{rank}(S) = 3$ (obviously S is not onto).

- For T we have... $$[T]_\gamma^\delta = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

  Labeling variables x_1, x_2, x_3, and x_4, we have the equations: $x_1 - x_2 = 0$ and $x_3 - x_4 = 0$. x_2 and x_4 are free, so let $x_2 = s$ and $x_4 = t$ we get:
  $$\begin{aligned} x_1 &= s \\ x_2 &= s \\ x_3 &= t \\ x_4 &= t \end{aligned} \qquad \text{so that...} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} t$$

  Thus, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{N}([T]_\gamma^\delta)$ which corresponds to: $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ (a basis for $\mathrm{Ker}(T)$). Therefore, T is not 1-1 and $\mathrm{nullity}(T) = 2$.
  Next, the first and third columns of our coordinate matrix are pivot columns so that $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Col}([T]_\gamma^\delta)$. These coordinate vectors correspond to $\{(1,0),(0,1)\}$ (the standard basis for $\mathbb{R}^2$). Therefore, $\mathrm{Range}(T) = \mathbb{R}^2$ and $\mathrm{rank}(T) = 2$.

- Finally, for $T \circ S$ we have... $$[T \circ S]_\beta^\delta = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  If we label variables x_1, x_2, and x_3, we see that the matrix says: $x_1 = 0$. Thus x_2 and x_3 are free, say $x_2 = s$ and $x_3 = t$ so we get:
  $$\begin{aligned} x_1 &= 0 \\ x_2 &= s \\ x_3 &= t \end{aligned} \qquad \text{thus...} \qquad \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t$$

  Therefore, $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{N}([T \circ S]_\beta^\delta)$. These coordinate vectors correspond to: $\{t, t^2\}$ which is a basis for $\mathrm{Ker}(T \circ S)$. From this we see that $T \circ S$ is not 1-1 and $\mathrm{nullity}(T \circ S) = 2$.

  Next, the first column of our coordinate matrix is the only pivot column so that $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Col}([T \circ S]_\beta^\delta)$. This coordinate vector corresponds to $\{(1,-1)\}$ which is a basis for $\mathrm{Range}(T \circ S)$. We see from this that $T \circ S$ is not onto since $\mathrm{rank}(T \circ S) = 1$ (< 2).