Let's explore coordinate representations of matrices and changes of basis.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x,y) = (3x+2y, 4y)$. We will consider the relationships between matrix representations of T in coordinate systems associated with the following bases:

- $\alpha = \{(1,0),(0,1)\}$ (The standard basis)
- $\beta = \{(1,1),(-2,3)\}$
- $\beta' = \{(1,-1),(0,1)\}$

First, let's compute $[T]_\alpha$ (the standard matrix of T), $[T]_\beta$, and $[T]_{\beta'}$.

- $T((1,0)) = (3,0) = 3(1,0)+0(0,1)$ so that $[T((1,0))]_\alpha = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and

  $T((0,1)) = (2,4) = 2(1,0)+4(0,1)$ so that $[T((0,1))]_\alpha = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Therefore,

$$[T]_\alpha = \begin{bmatrix} [T((1,0))]_\alpha & [T((0,1))]_\alpha \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix} = A$$

- $T((1,1)) = (5,4) = \frac{23}{5}(1,1) - \frac{1}{5}(-2,3)$ so that $[T((1,1))]_\beta = \begin{bmatrix} 23/5 \\ -1/5 \end{bmatrix}$ and

  $T((-2,3)) = (0,12) = \frac{24}{5}(1,1) + \frac{12}{5}(-2,3)$ so that $[T((-2,3))]_\beta = \begin{bmatrix} 24/5 \\ 12/5 \end{bmatrix}$. Therefore,

$$[T]_\beta = \begin{bmatrix} [T((1,1))]_\beta & [T((-2,3))]_\beta \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 23 & 24 \\ -1 & 12 \end{bmatrix} = B$$

- $T((1,-1)) = (1,-4) = 1(1,-1) - 3(0,1)$ so that $[T((1,-1))]_{\beta'} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and

  $T((0,1)) = (2,4) = 2(1,-1)+6(0,1)$ so that $[T((1,-1))]_{\beta'} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$. Therefore,

$$[T]_{\beta'} = \begin{bmatrix} [T((1,-1))]_{\beta'} & [T((0,1))]_{\beta'} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix} = C$$

Next, let's compute change of basis matrices. To convert from β-coordinates to α-coordinates and then from β'-coordinates to α-coordinates.

- $(1,1) = 1(1,0)+1(0,1)$ so that $[(1,1)]_\alpha = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

  $(-2,3) = -2(1,0)+3(0,1)$ so that $[(-2,3)]_\alpha = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Therefore, we can change from β to α coordinates with the following matrix:

$$[I]_\beta^\alpha = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} = P$$

- $(1,-1) = 1(1,0) - 1(0,1)$ so that $[(1,-1)]_\alpha = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and

  $(0,1) = 0(1,0) + 1(0,1)$ so that $[(0,1)]_\alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore, we can change from β' to α coordinates with the following matrix:

$$[I]_{\beta'}^{\alpha} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = Q$$

- To change from β to β' coordinates we can follow: $\beta \to \alpha \to \beta'$ thus

$$[I]_{\beta}^{\beta'} = [I]_{\alpha}^{\beta'}[I]_{\beta}^{\alpha} = Q^{-1}P = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

  (Or directly, $(1,1) = 1(1,-1) + 2(0,1)$ and $(-2,3) = -2(1,-1) + 1(0,1)$.)

Notice the following:

$$[T]_\beta = [I]_{\alpha}^{\beta}[T]_\alpha[I]_{\beta}^{\alpha} = P^{-1}AP = B$$

$$[T]_{\beta'} = [I]_{\alpha}^{\beta'}[T]_\alpha[I]_{\beta'}^{\alpha} = Q^{-1}AQ = C$$

$$[T]_{\beta}^{\beta'} = [I]_{\beta}^{\beta'}[T]_\beta = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 23 & 24 \\ -1 & 12 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 9 & 12 \end{bmatrix}$$

or

$$[T]_{\beta}^{\beta'} = [I]_{\alpha}^{\beta'}[T]_\alpha[I]_{\beta}^{\alpha} = Q^{-1}AP = \begin{bmatrix} 5 & 0 \\ 9 & 12 \end{bmatrix}$$

Finally, let $\mathbf{v} = (-1,3)$ then $[\mathbf{v}]_\alpha = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ so that $[\mathbf{v}]_{\beta'} = [I]_{\alpha}^{\beta'}[\mathbf{v}]_\alpha = Q^{-1}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Also, $[\mathbf{v}]_\beta = [I]_{\alpha}^{\beta}[\mathbf{v}]_\alpha = P^{-1}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$.

$$[T(\mathbf{v})]_{\beta'} = [T]_{\beta}^{\beta'}[v]_\beta = \begin{bmatrix} 5 & 0 \\ 9 & 12 \end{bmatrix}\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$$

On the other hand, $T(\mathbf{v}) = T((-1,3)) = (3,12)$ and $[T(\mathbf{v})]_{\beta'} = [I]_{\alpha}^{\beta'}\begin{bmatrix} 3 \\ 12 \end{bmatrix} = Q^{-1}\begin{bmatrix} 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$.