Let's explore the following linear transformation:

$$T : \mathbb{R}^{2\times 2} \to P_1 = \{ax+b \mid a,b \in \mathbb{R}\} \qquad \text{defined by} \qquad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+2b-c)x + (a+d)$$

T is linear: Let's prove T preserves addition and scalar multiplication.

$$T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) = ((a_1+a_2)+2(b_1+b_2)-(c_1+c_2))x+((a_1+a_2)+(d_1+d_2))$$

$$= ((a_1+2b_1-c_1)x+(a_1+d_1)) + ((a_2+2b_2-c_2)x+(a_2+d_2)) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$T\left(s\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}\right) = (sa+2sb-sc)x+(sa+sd) = s((a+2b-c)x+(a+d)) = s\,T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

Standard coordinate matrix: Let $\alpha = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and $\beta = \{1, x\}$.

These are the standard bases for $\mathbb{R}^{2\times 2}$ and P_1.

To find the coordinate matrix we plug each α (input basis) vector into our map: $T(E_{11}) = (1+2(0)-0)x+(1+0) = x+1$, $T(E_{12}) = (0+2(1)-0)x+(0+0) = 2x$, $T(E_{21}) = (0+2(0)-1)x+(0+0) = -x$, $T(E_{22}) = (0+2(0)-0)x+(0+1) = 1$. Then we write these in terms of β (output basis) coordinates (note the order of β – constant term then coefficient of x):

$$[T]_\alpha^\beta = \Big[\ [T(E_{11})]_\beta\ [T(E_{12})]_\beta\ [T(E_{21})]_\beta\ [T(E_{22})]_\beta\ \Big] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & -1 & 0 \end{bmatrix}$$

Standard basis of linear transformations: Keep the same bases α and β. The map that sends the j-th α (input basis) vector to the i-th β (output basis) vector and all other input vectors to $\mathbf{0}$ is called T_{ij}. In particular,

$$\begin{array}{lllll}
T_{11}(E_{11}) = 1, & T_{11}(E_{12}) = 0, & T_{11}(E_{21}) = 0, & \text{and} & T_{11}(E_{22}) = 0 \\
T_{12}(E_{11}) = 0, & T_{12}(E_{12}) = 1, & T_{12}(E_{21}) = 0, & \text{and} & T_{12}(E_{22}) = 0 \\
T_{13}(E_{11}) = 0, & T_{13}(E_{12}) = 0, & T_{13}(E_{21}) = 1, & \text{and} & T_{13}(E_{22}) = 0 \\
T_{14}(E_{11}) = 0, & T_{14}(E_{12}) = 0, & T_{14}(E_{21}) = 0, & \text{and} & T_{14}(E_{22}) = 1 \\
T_{21}(E_{11}) = x, & T_{21}(E_{12}) = 0, & T_{21}(E_{21}) = 0, & \text{and} & T_{21}(E_{22}) = 0 \\
T_{22}(E_{11}) = 0, & T_{22}(E_{12}) = x, & T_{22}(E_{21}) = 0, & \text{and} & T_{22}(E_{22}) = 0 \\
T_{23}(E_{11}) = 0, & T_{23}(E_{12}) = 0, & T_{23}(E_{21}) = x, & \text{and} & T_{23}(E_{22}) = 0 \\
T_{24}(E_{11}) = 0, & T_{24}(E_{12}) = 0, & T_{24}(E_{21}) = 0, & \text{and} & T_{24}(E_{22}) = x
\end{array}$$

These maps are rigged up so that $[T_{ij}]_\alpha^\beta = E_{ij}$. So just as $\{E_{11}, E_{12}, E_{13}, E_{14}, E_{21}, E_{22}, E_{23}, E_{24}\}$ is a basis for $\mathbb{R}^{2\times 4}$, we have that $\gamma = \{T_{11}, T_{12}, T_{13}, T_{14}, T_{21}, T_{22}, T_{23}, T_{24}\}$ is a basis for the space of linear transformations from $\mathbb{R}^{2\times 4}$ to P_1 (that is $\mathcal{L}(\mathbb{R}^{2\times 2}, P_1) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{2\times 2}, P_1)$).

For example, notice that $(T_{11}+T_{14}+T_{21}+2T_{22}-T_{23})(E_{11}) = T_{11}(E_{11})+T_{14}(E_{11})+T_{21}(E_{11})+2T_{22}(E_{11})-T_{23}(E_{11}) = 1+0+x+2(0)-0 = 1+x = T(E_{11})$. In fact, $T_{11}+T_{14}+T_{21}+2T_{22}-T_{23}$ and T match on all the α basis vectors. Therefore, $T = T_{11}+T_{14}+T_{21}+2T_{22}-T_{23}$.

Our coordinate isomorphisms are compatible with this as well. Notice that $[T]_\alpha^\beta = E_{11}+E_{14}+E_{21}+2E_{22}-E_{23}$ (changing the T_{ij}'s to $E'_{ij}s$).

Therefore, in γ coordinates we have…

$$T \text{ as a vector in } \mathcal{L}(\mathbb{R}^{2\times 2}, P_1): \quad [T]_\gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

…compared with…

$$T \text{ as a coordinate matrix:} \qquad [T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & -1 & 0 \end{bmatrix}$$

In general: Let V be an n-dimensional space (over $\mathbb{F}$) with basis $\alpha = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and let W be an m-dimensional space (over $\mathbb{F}$) with basis $\beta = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$. For each $1 \le k \le n$, define

$$T_{ij}(\mathbf{v}_k) = \delta_{jk}\mathbf{w}_i = \begin{cases} \mathbf{w}_i & j = k \\ \mathbf{0} & j \neq k \end{cases}$$

(this is the map that sends input basis vector j to output basis vector i and then kills the rest of the input vectors).

We have that $\gamma = \{T_{11}, \dots, T_{1n}, T_{21}, \dots, T_{2n}, \dots, T_{m1}, \dots, T_{mn}\}$ is a basis for $\mathcal{L}(V,W) = \mathrm{Hom}_{\mathbb{F}}(V,W)$ (the space of all linear maps from V to W).

Let $T : V \to W$ be a linear map (i.e. $T \in \mathcal{L}(V,W) = \mathrm{Hom}_{\mathbb{F}}(V,W)$). Let $A = (a_{ij}) = [T]_\alpha^\beta$. This means that $T(\mathbf{v}_k) = \sum_{i=1}^m a_{ik}\mathbf{w}_i$ (the k-th column of $A = [T]_\alpha^\beta$ is given by the coordinates of $T(\mathbf{v}_k)$).

Consider $S = \sum_{i=1}^m \sum_{j=1}^n a_{ij}T_{ij}$. Then $S(\mathbf{v}_k) = \sum_{i=1}^m \sum_{j=1}^n a_{ij}T_{ij}(\mathbf{v}_k) = \sum_{i=1}^m \sum_{j=1}^n a_{ij}\delta_{jk}\mathbf{w}_i = \sum_{i=1}^m a_{ik}\mathbf{w}_i$ (only the $j = k$ terms survive). Therefore, $S(\mathbf{v}_k) = T(\mathbf{v}_k)$ for $1 \le k \le n$. Since S and T match on a basis, $S = T$. We have that $T = \sum_{i=1}^m \sum_{j=1}^n a_{ij}T_{ij}$. In other words, the γ-coordinates of T are exactly the entries of $A = [T]_\alpha^\beta$ (its coordinate matrix).

$$\text{If} \quad [T]_\alpha^\beta = A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{then} \quad [T]_\gamma = \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \\ a_{21} \\ \vdots \\ a_{2n} \\ \vdots \\ a_{m1} \\ \vdots \\ a_{mn} \end{bmatrix}.$$