Math 5230 # Homework #5 **Due: Mon., Oct. 10^{th}, 2016**

Notation: V is a vector space over a field $\mathbb{F}$. Do not assume V is finite dimensional (unless told otherwise). Let W_1 and W_2 be subspaces of V.

#1 Vector Space Definition Using the definition – and that's all.

(a) In horrifying detail (go through all of the axioms), show that $\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$.

(b) Let $\mathbf{u},\mathbf{v},\mathbf{w} \in V$. Prove that $\mathbf{u}+\mathbf{v} = \mathbf{u}+\mathbf{w}$ implies that $\mathbf{v} = \mathbf{w}$. Do this *only using axioms.* Cite the axioms being used.

#2 Easy Subspace Show that $W = \{(x,y) \in \mathbb{R}^2 \mid x+2y=0\}$ is a subspace of $\mathbb{R}^2$. Then explain why $B = \{(x,y) \in \mathbb{R}^2 \mid x+2y=3\}$ is not a subspace.

#3 Subspace Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called **even** if $f(-x) = f(x)$. A function is called **odd** if $f(-x) = -f(x)$. Recall that $V = \mathbb{R}^{\mathbb{R}}$ is the vector space of all functions from the reals to the reals.

(a) Let E be the set of all even functions in $\mathbb{R}^{\mathbb{R}}$. Show that E is a subspace.

(b) Let O be the set of all odd functions in $\mathbb{R}^{\mathbb{R}}$. Show that O is a subspace.

(c) Notice that for any $f \in \mathbb{R}^{\mathbb{R}}$, $g(x) = \dfrac{f(x)+f(-x)}{2}$ is even and $h(x) = \dfrac{f(x)-f(-x)}{2}$ is odd. Prove that $\mathbb{R}^{\mathbb{R}} = E \oplus O$. Demonstrate how $f(x) = x^3 - 5x^2 + \cos(x) + 4\sin(x) - 12$ decomposes.

"Fun" Side Note: When we decompose a function into an even function plus an odd function, we call these parts it's even and odd parts. The even and odd parts of the exponential function, e^x, are called $\cosh(x)$ and $\sinh(x)$ (the hyperbolic cosine and sine functions).

#4 Unions Don't Work Show that $W_1 \cup W_2$ is a subspace if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

#5 Linear Independence and Spanning Let V be a vector space over $\mathbb{F} = \mathbb{Q}$ (the rational numbers) with basis $\beta = \{\mathbf{u},\mathbf{v},\mathbf{w}\}$ (distinct vectors).

(a) Prove that $S = \{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}\}$ is linearly independent (using the definition of independence).

(b) Prove that $T = \{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{w}\}$ spans V (using the definition of span).

(c) Consider $S \cup T$ and $S \cap T$. Decide whether each set is linearly independent, spans V, both spans and is linearly independent, or neither. Give proof of each assertion.

#6 Dimension For convenience, let's assume V is finite dimensional. Note that $W_1 + W_2 = \{u+v \mid u \in W_1 \text{ and } v \in W_2\}$. Also, recall that $V = W_1 \oplus W_2$ if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\mathbf{0}\}$.

(a) Prove that $\dim(W_1 \cap W_2) \le \min\{\dim(W_1), \dim(W_2)\}$.

(b) Prove that $\dim(W_1 + W_2) \ge \max\{\dim(W_1), \dim(W_2)\}$.

(c) If $V = W_1 \oplus W_2$, what is $\dim(V)$? Justify your answer.

As a general bit of advice for problems like this, assume (without loss of generality) that $\dim(W_1) = m \le \dim(W_2) = n$ so that (for example) the minimum of the two dimensions is m and the maximum is n.