Math 5230 # Homework #6 **Due: Wed., Oct. 26^{th}, 2016**

#1 Bases Consider $W = \left\{ \begin{bmatrix} -3a-4b+c & b-c & -4a-6b+2c \\ a+b & 2b-2c & a+b \end{bmatrix} \;\middle|\; a,b,c \in \mathbb{R} \right\}$ and

$$V = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \;\middle|\; \begin{array}{rcl} a+3b+2c+4d+e+7f &=& 0 \\ 2a+6b+c+5d-e+5f &=& 0 \\ 3a+9b+c+7d-2e+6f &=& 0 \end{array} \right\}.$$

(a) Both W and V are subspaces of $\mathbb{R}^{2\times 3}$. Prove that they are subspaces by re-expressing each as either a span or kernel [this would prove they are subspaces since kernels and spans are always subspaces]. Pick the most convenient description.

(b) Prove that $W \subseteq V$.

(c) Find a basis for V.

(d) Find a basis for W and then extend this to a basis for all of V.

 # 2 Linear Transformation Basics Define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}[x]$ as follows:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+2b+3d)x^2 + (-a-2b+c+d)x + (a+2b+c+7d).$$

Note that T is a linear transformation (you don't need to prove this).

(a) Write down the standard coordinate matrix for T and find its RREF.
[You may truncate the infinitely many rows of zeros.]

(b) Find a basis for the kernel and range of T.

(c) What is the nullity and rank of T? Is T 1-1, onto, both, neither?

 # 3 Trace Let $A \in \mathbb{F}^{n\times n}$ ($\mathbb{F}$ is a field) and let a_{ij} be the (i,j)-entry of A. Recall that $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$ (the *trace* of A is the sum of its diagonal entries).

(a) Prove that $\mathrm{tr} : \mathbb{F}^{n\times n} \to \mathbb{F}$ is linear.

(b) Identify the kernel, range, nullity, and rank of the trace map.

#4 Linear Let $\mathcal{I}, \mathcal{D} : \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by $\mathcal{I}(f(x)) = \int_0^x f(t)\,dt$ and $\mathcal{D}(f(x)) = f'(x)$.

(a) Show $\mathcal{I}$ is linear. Identify its kernel and range. Is $\mathcal{I}$ one-to-one? onto? an isomorphism?

(b) Show $\mathcal{D}$ is linear. Identify its kernel and range. Is $\mathcal{D}$ one-to-one? onto? an isomorphism?

(c) Describe $\mathcal{I} \circ \mathcal{D}$ and $\mathcal{D} \circ \mathcal{I}$.

#5 Quotient Let U and W be subspaces of some vector space V (over a field $\mathbb{F}$).

(a) Prove that $W/(U \cap W) \cong (U+W)/U$.

(b) Give a relationship among the dimensions of $U, W, U+W$, and $U \cap W$ determined by the above isomorphism.

(c) What can we say in the special case when $U + W = U \oplus W$?