**Math 5230** # Homework #7 **Due: Fri., Nov. $11^{\text{th}}$, 2016**

**#1 Linearity** Let $T : V \to W$ and $S : W \to U$ be linear transformations between vector spaces $V$, $W$, and $U$ (over some field $\mathbb{F}$).

(a) Prove that $S \circ T$ is linear.

(b) Assume that $T$ is a bijection. Show that $T^{-1}$ is linear (i.e. if $T$ is an isomorphism, then so is $T^{-1}$).

**#2 Combating Coordinate Change** Let $T : P_2 \to \mathbb{R}^{2\times 2}$ and $S : \mathbb{R}^{2\times 2} \to \mathbb{R}^2$ be defined by

$$T(ax^2 + bx + c) = \begin{bmatrix} a+b+c & 2a-b+5c \\ b-c & a+2b \end{bmatrix} \quad \text{and} \quad S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b-c, b+c+d).$$

Let $\alpha = \{1, x, x^2\}$, $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, and $\gamma = \{(1,0),(0,1)\}$ be the standard bases for $P_2$, $\mathbb{R}^{2\times 2}$, and $\mathbb{R}^2$ respectively.

(a) Show $S$ and $T$ are linear. [So by the first problem, we have that $S \circ T$ is linear too.]

(b) Find $[T]_\alpha^\beta$, $[S]_\beta^\gamma$, and $[S \circ T]_\alpha^\gamma$.

(c) Let $\delta = \{x, 1+x, x+x^2\}$. Explain why $\delta$ is a basis for $P_2$ and then find the change of basis matrices $[I]_\delta^\alpha$ and $[I]_\alpha^\delta$.

(d) Let $\kappa = \{(1,2),(3,4)\}$. Explain why $\kappa$ is a basis for $\mathbb{R}^2$ and then find the change of basis matrices $[I]_\kappa^\gamma$ and $[I]_\gamma^\kappa$.

(e) Compute $[S \circ T]_\delta^\kappa$.

**#3 Dual Basis** Let $\alpha = \{(1,0,0),(1,-1,0),(2,0,1)\}$.

(a) Explain why $\alpha$ is a basis for $\mathbb{R}^3$.

(b) Find $\alpha^*$ for $(\mathbb{R}^3)^*$ (i.e. find the basis dual to $\alpha$).

(c) Explain why $f \in (\mathbb{R}^3)^*$ where $f(x,y,z) = 3x+2y+z$. Then write $f$ as a linear combination of $\alpha^*$ elements (i.e. find its $\alpha^*$-coordinates).

**#4 Dual Proof** Let $W$ be a subspace of a vector space $V$ (over a field $\mathbb{F}$). We say that $f \in V^*$ **annihilates** $W$ if $f(\mathbf{w}) = 0$ for all $\mathbf{w} \in W$. Let $A(W) = \{f \in V^* \mid f(\mathbf{w}) = 0 \text{ for all } \mathbf{w} \in W\}$ (the collection of all linear functionals which annihilate $W$). $A(W)$ is called the **annihilator** of $W$.

(a) Prove that $A(W)$ is a subspace of $V^*$.

(b) Suppose that $V = U \oplus W$ for some subspaces $U$ and $W$. Show that $V^* = A(W) \oplus A(U)$.

(c) Let $T : V \to V$ be a linear operator and suppose that $T(W) \subseteq W$ for some subspace $W$ (i.e. $W$ is a $T$-invariant subspace). Show that $T^t(A(W)) \subseteq A(W)$ (i.e. $A(W)$ is $T^t$-invariant).