Math 5230 # Homework #5 **Due: Fri., Oct. 7^{th}, 2022**

Notation: V is a vector space over a field $\mathbb{F}$. Do not assume V is finite dimensional (unless told otherwise).

#1 Vector Space Definition Building up from nothing. I'll provide a few example axiom level proofs.

The zero vector is unique: Suppose $\mathbf{0}$ and $\mathbf{0}'$ are zero vectors. Then $\mathbf{0}' = \mathbf{0} + \mathbf{0}' = \mathbf{0}$ where the first equality follows from the fact the adding the zero vector $\mathbf{0}$ does nothing and the second equality follows from the fact that adding the zero vector $\mathbf{0}'$ also does nothing.

Additive inverses are unique: Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Suppose that $\mathbf{u}$ is a left and $\mathbf{w}$ is a right additive inverse of $\mathbf{v}$. Then $\mathbf{w} = \mathbf{0} + \mathbf{w} = (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) = \mathbf{u} + \mathbf{0} = \mathbf{u}$ where we used the definition of the zero vector, the definitions of left and right additive inverses, and associativity of vector addition to get our equalities.

The scalar zero yields zero: Let $\mathbf{v} \in V$. Then $0\mathbf{v} = (0+0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$ since $0+0=0$ (the scalar zero is the additive identity in $\mathbb{F}$). We also used the fact that we can distribute scalar multiplication. Now add $-(0\mathbf{v})$ to both sides: $-(0\mathbf{v}) + 0\mathbf{v} = -(0\mathbf{v}) + (0\mathbf{v} + 0\mathbf{v})$. Thus $\mathbf{0} = (-(0\mathbf{v}) + 0\mathbf{v}) + 0\mathbf{v} = \mathbf{0} + 0\mathbf{v} = 0\mathbf{v}$ where we used the definition of additive inverses, associativity, and additive identity properties. Therefore, $0\mathbf{v} = \mathbf{0}$.

Negative one negates: Let $\mathbf{v} \in V$. Then $\mathbf{0} = 0\mathbf{v} = (1-1)\mathbf{v} = 1\mathbf{v} + (-1)\mathbf{v} = \mathbf{v} + (-1)\mathbf{v}$ where we have used our result immediately above, the definition of -1, distribution, and the axiom that scaling by 1 does nothing. Now add $-\mathbf{v}$ to both side: $-\mathbf{v} + \mathbf{0} = -\mathbf{v} + (\mathbf{v} + (-1)\mathbf{v})$. Using the additive identity property of the zero vector and associativity we get: $-\mathbf{v} = (-\mathbf{v} + \mathbf{v}) + (-1)\mathbf{v} = \mathbf{0} + (-1)\mathbf{v}$. Therefore, $(-1)\mathbf{v} = -\mathbf{v}$.

(a) In horrifying detail (go through all of the axioms), show $\mathbb{R}^2 = \{(a,b) \mid a, b \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$.

(b) [Cancellation:] Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Prove that $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$ implies that $\mathbf{v} = \mathbf{w}$.
Do this *only using axioms*. Cite the axioms being used.

(c) [Scaling zero does nothing:] Let $c \in \mathbb{F}$ and $\mathbf{0}$ be the zero vector in V. Prove $c\mathbf{0} = \mathbf{0}$.
Carefully explain each step.

#2 Easy Subspace Use the subspace test.

(a) Show that $W = \{(x,y) \in \mathbb{R}^2 \mid x + 2y = 0\}$ is a subspace of $\mathbb{R}^2$.

(b) Explain why $B = \{(x,y) \in \mathbb{R}^2 \mid x + 2y = 3\}$ is not a subspace of $\mathbb{R}^2$.

#3 Trickier Subspace The collection of all functions from a field to itself, $V = \mathbb{F}^{\mathbb{F}} = \{f \mid f : \mathbb{F} \to \mathbb{F}\}$ is a vector space (over $\mathbb{F}$).

Let $f \in V$. We say that f is **even** if $f(-x) = f(x)$ for all $x \in \mathbb{F}$. Likewise, we say that f is **odd** if $f(-x) = -f(x)$. For example, $f(x) = x^2$ is even whereas $g(x) = x^3$ is odd. On the other hand, generally $h(x) = x^2 + x^3$ is neither.

(a) Let $E = \{f \in V \mid f \text{ is even }\}$. Show E is a subspace of V.

(b) Let $O = \{f \in V \mid f \text{ is odd }\}$. Show O is a subspace of V.

(c) Suppose that $\mathbb{F}$ is a field such that $\text{char}(\mathbb{F}) \neq 2$ (i.e., 2^{-1} exists).

For any $f \in V$, let $f_e(x) = \dfrac{f(x) + f(-x)}{2}$ and $f_o(x) = \dfrac{f(x) - f(-x)}{2}$ define functions f_e and f_o in V.

Notice: $\quad f_e(-x) = \dfrac{f(-x) + f(x)}{2} = f_e(x) \quad$ and $\quad f_o(-x) = \dfrac{f(-x) - f(x)}{2} = -\dfrac{f(x) - f(-x)}{2} = -f_o(x)$

Therefore, $f_e \in E$ and $f_o \in O$.

Show that $V = E \oplus O$. [Thus every function can be uniquely decomposed into an even plus an odd part.]

Note: Showing V is a direct sum of E and O requires two things. First, you need to show that every $f \in V$ can be written as $f = g + h$ where $g \in E$ and $h \in O$. Then, you need to show that $E \cap O = \{0\}$ where 0 here is the zero function: $0(x) = 0$ for all $x \in \mathbb{F}$. The first part shows that $V = E + O$ and the second part shows that the sum is *direct*.

(d) Working over $\mathbb{F} = \mathbb{R}$, how does $f(x) = x^3 - 5x^2 + \cos(x) + 4\sin(x) - 12$ decompose?

Side Notes:

- In characteristic 2, $x = -x$ so $-f(x) = f(x) = f(-x)$. This means $V = E = O$.
- Again, working over the reals, the even part of e^x is $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ and the odd part is $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ (the hyperbolic sine and cosine functions).

#4 Unions Don't Work Let W_1 and W_2 be subspaces of V.

Show that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

#5 Linear Independence and Spanning Suppose V has a basis $\beta = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ (distinct vectors).

Let $S = \{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}\}$ and $T = \{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{w}\}$.

Additional Assumption: $\text{char}(\mathbb{F}) \neq 2$.

(a) Prove S is linearly independent (using the definition of independence).

(b) Prove T spans V (using the definition of span).

(c) Is $S \cup T$ linearly independent? Does it span V? Explain.

(d) Is $S \cap T$ linearly independent? Does it span V? Explain.