#1 Bases Consider $W = \left\{ \begin{bmatrix} -3a - 4b + c & b - c & -4a - 6b + 2c \\ a + b & 2b - 2c & a + b \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$ and

$$V = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \;\middle|\; \begin{array}{rcl} a + 3b + 2c + 4d + e + 7f & = & 0 \\ 2a + 6b + c + 5d - e + 5f & = & 0 \\ 3a + 9b + c + 7d - 2e + 6f & = & 0 \end{array} \right\}.$$

(a) Both W and V are subspaces of $\mathbb{R}^{2\times3}$. Prove that they are subspaces by re-expressing each as either a span or kernel (i.e., null space). Pick the most convenient description.

Note: Re-expressing as a span or kernel would prove we have a subspace since spans and kernels are always subspaces.

(b) Prove that $W \subseteq V$.

(c) Find a basis for V.

(d) Find a basis for W and then extend this to a basis for all of V.

#2 Linear Transformation Basics Define $T : \mathbb{R}^{2\times2} \to \mathbb{R}[x]$ as follows:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + 2b + 3d)x^2 + (-a - 2b + c + d)x + (a + 2b + c + 7d).$$

Note that T is a linear transformation (you don't need to prove this).

(a) Write down the standard coordinate matrix for T and find its RREF.
[You may truncate the infinitely many rows of zeros.]

(b) Find a basis for the kernel and range of T.

(c) What is the nullity and rank of T? Is T 1-1, onto, both, neither?

#3 Trace Let $A \in \mathbb{F}^{n\times n}$ ($\mathbb{F}$ is a field) and let a_{ij} be the (i,j)-entry of A. Recall that $\operatorname{tr}(A) = \sum\limits_{i=1}^{n} a_{ii}$ (the *trace* of A is the sum of its diagonal entries).

(a) Prove that $\operatorname{tr} : \mathbb{F}^{n\times n} \to \mathbb{F}$ is linear.

(b) Identify the kernel, range, nullity, and rank of the trace map.

#4 Linear Let $\mathcal{I}, \mathcal{D} : \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by $\mathcal{I}(f(x)) = \int_0^x f(t)\, dt$ and $\mathcal{D}(f(x)) = f'(x)$.

(a) Show $\mathcal{I}$ is linear. Identify its kernel and range. Is $\mathcal{I}$ one-to-one? onto? an isomorphism?

(b) Show $\mathcal{D}$ is linear. Identify its kernel and range. Is $\mathcal{D}$ one-to-one? onto? an isomorphism?

(c) Describe $\mathcal{I} \circ \mathcal{D}$ and $\mathcal{D} \circ \mathcal{I}$.