Math 5230 # Homework #7 **Due: Fri., Oct. 28^{th}, 2022**

#1 Ultimately a Based Problem Let $T : V \to W$ be a linear transformation between vector spaces V and W (both over the field $\mathbb{F}$).

> Let S be a linearly independent subset of V and let T be one-to-one. Show $T(S)$ is linearly independent.

Note: We have $T\left(\sum_{i=1}^{\ell} c_i\mathbf{v}_i\right) = \sum_{i=1}^{\ell} c_iT(\mathbf{v}_i)$ for any $c_1, \dots, c_\ell \in \mathbb{F}$ and $\mathbf{v}_1, \dots, \mathbf{v}_\ell \in V$. Thus the image of any linear combination of elements of V is a linear combination of the images of those elements. Therefore, given any subset $S \subseteq V$, we have $T(\text{span}(S)) = \text{span}(T(S))$. Consequently, if S spans V, then $T(S)$ spans $T(V)$. Thus if T is onto, then a spanning set for V maps to a spanning set for W.

Putting this together with the above homework problem, we get that **isomorphisms map bases to bases**.

#2 Concrete Quotient Let $W = \left\{ \begin{bmatrix} a+b+4c & 2a+b+3c \\ 3a+b+2c & 4a+b+c \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$. Give a quick justification for why W is a *subspace* of $\mathbb{R}^{2\times 2}$. Then find a basis for W and a basis for $\mathbb{R}^{2\times 2}/W$.

#3 Abstract Quotient Let U and W be subspaces of some vector space V (over a field $\mathbb{F}$).

(a) Prove the Second (or Diamond) Isomorphism Theorem: $W/U \cap W \cong U + W/U$.

Hint: Consider $\varphi : W \to U + W/U$ defined by $\varphi(\mathbf{w}) = \mathbf{w} + U$ and apply the First Isomorphism Theorem.

(b) Give a relationship among the dimensions of $U, W, U + W$, and $U \cap W$ determined by the above theorem

(c) What can we say in the special case when $U + W = U \oplus W$?

#4 Concrete Dual Let $\alpha = \{(1, 0, 0), (1, -1, 0), (2, 0, 1)\}$.

(a) Explain why α is a basis for $\mathbb{R}^3$. Then find α^* for $(\mathbb{R}^3)^*$ (i.e. find the basis dual to α).

(b) Explain why $f \in (\mathbb{R}^3)^*$ where $f(x, y, z) = 3x+2y+z$. Then write f as a linear combination of α^* elements (i.e., find its α^*-coordinates).

#5. Completely Annihilated Let V be a vector space over $\mathbb{F}$ and let W be a subspace of V. We define $A(W) = \{f \in V^* \mid f(w) = 0 \text{ for all } w \in W\}$. In other words, $f \in A(W)$ if $f(W) = \{0\}$ (i.e., f annihilates all of our subspace W).

(a) Show that $A(W)$ is a subspace of V^*. [I'll do this one for you.]
First, the zero functional sends all vectors to zero (the scalar). Thus $0 \in A(W)$ (i.e., the annihilator of W is a non-empty subset of V^*). Let $f, g \in A(W)$ and $s \in \mathbb{F}$. Notice that for all $\mathbf{w} \in W$, we have $(f + g)(\mathbf{w}) = f(\mathbf{w}) + g(\mathbf{w}) = 0 + 0 = 0$ and $(sf)(\mathbf{w}) = s\,f(\mathbf{w}) = s\,0 = 0$. Thus $f + g, sf \in A(W)$. Therefore, $A(W)$ is a subspace.

(b) Suppose U is a subspace of W. Explain why $A(W) \subseteq A(U)$. [And this one too.]
Let $f \in A(W)$. This means that $f(\mathbf{w}) = 0$ for all $\mathbf{w} \in W$. Suppose $\mathbf{u} \in U$. Then because $U \subseteq W$ we have $\mathbf{u} \in W$ and so $f(\mathbf{u}) = 0$. Therefore, f annihilates all of U and thus $f \in A(U)$. Thus $A(W) \subseteq A(U)$. Briefly, if we annihilate all of W, then since U is contained in W, we certainly annihilate all of U.

(c) Suppose $V = U \oplus W$. Show that $V^* = A(W) \oplus A(U)$.

Note: You need to show that every dual vector is a sum of a dual vector annihilating W and one annihilating U. Also, you need to show that if $f \in A(W) \cap A(U)$ then $f = 0$.

Big hint: Consider $\pi_U : V \to V$ defined by $\pi(\mathbf{u} + \mathbf{w}) = \mathbf{u}$ where $\mathbf{u} \in U$ and $\mathbf{w} \in W$ (this is well defined since every vector in V is a *unique* sum of a vector in U and a vector W – because V is a *direct sum* of those spaces). This π_U is called a projection onto U. It is linear. Likewise, define π_W. Consider composing $f \in V^*$ with these maps.

(d) Let $T : V \to V$ be a linear operator and suppose that $T(W) \subseteq W$ (i.e., W is T-invariant). Show that $T^*(A(W)) \subseteq A(W)$ (i.e., $A(W)$ is T^*-invariant).

Recall: $T^* : V^* \to V^*$ is the transpose of T defined by $T^*(f) = f \circ T$.