Note: This example is less about linear algebra and more about cardinal numbers. Don't take it too seriously.

Theorem: The dimension of the space of continuous function is continuum.

Proof: First, some notation. Let $\mathcal{C} = \mathcal{C}^0 = \mathcal{C}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$ be the set of continuous functions (from the real numbers to themselves). This is a vector space over $\mathbb{R}$ since adding and scaling continuous functions yields continuous functions. Next, let $\mathcal{C}^k = \mathcal{C}^k(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f^{(k)} \text{ exists and is continuous}\}$. This is the space of k-times continuously differentiable functions. Also, let $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f^{(k)} \text{ exists for all } k \geq 0\}$. This is the space of smooth functions.

By definition, $f^{(k+1)}$ (the $(k+1)^{\text{st}}$ derivative of f) exists only if $f^{(k)}$ (the k^{th} derivative of f) exists. Moreover, a basic result from Calculus tells us that differentiable functions must be continuous functions. In particular, if $f^{(k+1)}$ exists, by definition $f^{(k)}$ is differentiable and thus must be continuous. Therefore, for any positive integer k, $\mathcal{C}^\infty \subseteq \cdots \subseteq \mathcal{C}^k \subseteq \cdots \subseteq \mathcal{C}^1 \subseteq \mathcal{C}^0$ $(= \mathcal{C})$. These are non-empty since the constant function 0 is smooth (and thus belongs to all of these sets). In addition, if $f^{(k)}$ and $g^{(k)}$ are continuous, then $(f+g)^{(k)}$ and $cf^{(k)}$ (for any $c \in \mathbb{R}$) are continuous as well. Therefore, these sets are (nested) subspaces of the space of all functions from the reals to the reals.

Now we consider cardinalities. We denote countable infinity by $\aleph_0$ and continuum infinity by $\mathfrak{c} = 2^{\aleph_0}$. A basic result from analysis says the rational numbers, $\mathbb{Q}$, are dense in the reals, $\mathbb{R}$. Another basic result tells us that a continuous function is determined by its values on any dense subset of its domain. Thus continuous functions are determined by their rational inputs. Therefore, there are at most $|\{f : \mathbb{Q} \to \mathbb{R}\}| = |\mathbb{R}|^{|\mathbb{Q}|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \mathfrak{c}$ (i.e., continuum cardinality) continuous functions. On the other hand, constant functions: $f(t) = c$ for any fixed $c \in \mathbb{R}$ are infinitely differentiable (e.g., $f^{(k)}(t) = 0$ for $k > 0$), so that there are at least $|\mathbb{R}| = \mathfrak{c}$ smooth functions. This means $\mathfrak{c} \leq |\mathcal{C}^\infty| \leq \cdots \leq |\mathcal{C}^k| \leq \cdots \leq |\mathcal{C}| \leq \mathfrak{c}$. Therefore, all of the above spaces (i.e., $\mathcal{C}$, $\mathcal{C}^\infty$, $\mathcal{C}^k$ for $k = 1, 2, \dots$) have cardinality $\mathfrak{c}$. Since vector spaces span themselves and every spanning set contains a basis: $\dim(V) \leq |V|$. In particular, the dimensions of these spaces are bounded above by their cardinality (i.e., $\mathfrak{c}$).

Finally, consider distinct real numbers $r_1, \dots, r_\ell \in \mathbb{R}$ and let $f_i(t) = e^{r_i t}$ for $i = 1, \dots, \ell$. Obviously, $f_i \in \mathcal{C}^\infty$. In fact, $f_i^{(k)}(t) = r_i^k e^{r_i t}$. Suppose $c_1 f_1 + \cdots + c_\ell f_\ell = 0$. Then $c_1 f_1^{(k)} + \cdots + c_\ell f_\ell^{(k)} = 0$ for any non-negative integer k. Evaluating at $t = 0$, we get $c_1 r_1^k e^{r_1 0} + \cdots + c_\ell r_\ell^k e^{r_\ell 0} = 0$ and thus $c_1 r_1^k + \cdots + c_\ell r_\ell^k = 0$. In particular, using the first k equations, we have:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_\ell \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{k-1} & r_2^{k-1} & \cdots & r_\ell^{k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_\ell \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The above coefficient matrix is called a Vandermonde matrix. It is well known that its determinant is $\prod_{i<j}^{\ell} (r_j - r_i) \neq 0$. Since our coefficient matrix is non-singular, the only solution to this system is the trivial solution: $c_1 = c_2 = \cdots = c_\ell = 0$. Consequently our functions are linearly independent.

We have shown that $S = \{e^{rt} \mid r \in \mathbb{R}\}$ is a linearly independent set of smooth functions. Since linearly independent sets can be extended to bases, $\mathfrak{c} = |S| \leq \dim(\mathcal{C}^\infty) \leq \cdots \leq \dim(\mathcal{C}^k) \leq \cdots \leq \dim(\mathcal{C})$. However, we already know these dimensions are bounded above by $\mathfrak{c}$. Therefore, $\dim(\mathcal{C}^\infty) = \dim(\mathcal{C}^k) = \dim(\mathcal{C}) = \mathfrak{c}$ (i.e., all of the function spaces under consideration are continuum-dimensional vector spaces over the real numbers). ■

Analytic Functions: If we let $\mathcal{C}^\omega = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is analytic }\}$ denote all real analytic functions (these are functions which can be expanded into Taylor series). Then one easily checks that $\mathcal{C}^\omega$ is a subspace of $\mathcal{C}^\infty$. Since the constant functions are analytic and e^{rt} is analytic for any $r \in \mathbb{R}$, the argument above shows that $\dim(\mathcal{C}^\omega) = \mathfrak{c}$ as well. In other words, the space of power series (= real analytic functions) is continuum dimensional. On the other hand, $\{1, t, t^2, \dots\}$ forms a basis for real polynomial functions so that it is only countable (i.e., $\aleph_0$) dimensional.

Note: Like the previous page, this example is weird in terms of our usual undertakings. Don't take it too seriously.

Let $\mathbb{F}$ be a field. Recall that $\mathbb{F}[t] = \{a_nt^n + \cdots + a_1t + a_0 \mid n \geq 0 \;;\; a_0, \ldots, a_n \in \mathbb{F}\}$ is the set of all polynomials in indeterminate t with coefficients in $\mathbb{F}$. This collection has the structure of a (commutative associative unital) *algebra* (over $\mathbb{F}$). Essentially this means one can add, scale (by $\mathbb{F}$), and multiply these polynomials. In terms of ring theory, $\mathbb{F}[t]$ is quite a nice of ring. It is a *Euclidean domain* (we can divide with remainder and the extended Euclidean algorithm works). This implies it is also a *principal ideal domain* (PID) and thus a *unique factorization domain* (UFD). Ignoring polynomial multiplication, $\mathbb{F}[t]$ is an infinite dimensional vector space over $\mathbb{F}$ with standard basis $\beta = \{1, t, t^2, \ldots\}$. This means $\dim(\mathbb{F}[t]) = \aleph_0$ (i.e., it is countably infinite dimensional).

Given an integral domain (a commutative ring where cancellation works), one can formally construct its *field of fractions*. For polynomials $\mathbb{F}[t]$, this would be $\mathbb{F}(t) = \{f(t)/g(t) \mid f(t), g(t) \in \mathbb{F}[t] \text{ and } g(t) \neq 0\}$. This ring $\mathbb{F}(t)$ is called the field of *rational functions* with coefficients in $\mathbb{F}$. Just like $\mathbb{F}[t]$, $\mathbb{F}(t)$ is a (commutative associative unital) algebra (over $\mathbb{F}$). While it is (just like $\mathbb{F}[t]$) a Euclidean domain, PID, UFD, and vector space (over $\mathbb{F}$), it is even nicer. $\mathbb{F}(t)$ is a field. We ask, what is $\dim(\mathbb{F}(t)$ (when working over $\mathbb{F}$)?

Let us sketch out how this works over a general field and then specialize to when $\mathbb{F}$ is the real or complex numbers. We recall $f(t) \in \mathbb{F}[t]$ is *monic* if its leading coefficient is 1, and $f(t)$ is *irreducible* if $f(t)$ is non-constant and has no non-trivial factorizations (i.e., $f(t) = g(t)h(t)$ implies $g(t)$ or $h(t)$ is constant). Because $\mathbb{F}[t]$ is a unique factorization domain, every non-constant polynomial can be factored uniquely (up to order and associates) into irreducibles.

Consider $\frac{f(t)}{g(t)} \in \mathbb{F}(t)$. Suppose $g(t) = a(t)b(t)$ where $a(t), b(t) \in \mathbb{F}[t]$ are relatively prime. By the extended Euclidean algorithm, there exists $m(t), n(t) \in \mathbb{F}[t]$ such that $m(t)a(t) + n(t)b(t) = 1$. Thus $\frac{f(t)}{g(t)} = \frac{f(t)(m(t)a(t)+n(t)b(t))}{a(t)b(t)} = \frac{f(t)m(t)}{b(t)} + \frac{f(t)n(t)}{a(t)}$. From this we learn if $g(t) = p_1(t)^{k_1} \cdots p_\ell(t)^{k_\ell}$ where $p_i(t)$ are non-associate (i.e., distinct and thus relatively prime) irreducibles and $k_i > 0$, then we can find $h_1(t), \ldots, h_\ell(t) \in \mathbb{F}[t]$ such that $\frac{f(t)}{g(t)} = \frac{h_1(t)}{p_1(t)^{k_1}} + \cdots + \frac{h_\ell(t)}{p_\ell(t)^{k_\ell}}$. Focus on a factor $\frac{h(t)}{p(t)^k}$. Because $\mathbb{F}[t]$ is a Euclidean domain, we have a division algorithm and can divide $h(t)$ by $p(t)$. We have unique $q(t), r(t) \in \mathbb{F}[t]$ such that $h(t) = p(t)q(t) + r(t)$ with $r(t) = 0$ or $\deg(r(t)) < \deg(p(t))$. Thus $\frac{h(t)}{p(t)^k} = \frac{q(t)}{p(t)^{k-1}} + \frac{r(t)}{p(t)^k}$. Continuing in this fashion, we can write $\frac{h(t)}{p(t)^k} = q(t) + \frac{r_1(t)}{p(t)} + \frac{r_2(t)}{p(t)^2} + \cdots + \frac{r_k(t)}{p(t)^k}$ where $q(t), r_1(t), \ldots, r_k(t) \in \mathbb{F}[t]$ and each $r_i(t)$ is either 0 or has degree at most $\deg(p(t))$. Putting all of this together, $\boxed{\frac{f(t)}{g(t)} = q(t) + \sum_{i=1}^{\ell} \sum_{j=1}^{k_i} \frac{r_{ij}(t)}{p_i(t)^j}}$ for some $q(t), r_{ij}(t) \in \mathbb{F}[t]$ where $r_{ij}(t) = 0$ or $\deg(r_{ij}(t)) < \deg(p_i(t))$ for all i, j. The boxed in equation is called the *partial fraction decomposition* of $\frac{f(t)}{g(t)}$.

It can be shown (see [BC]) this decomposition is *unique*. Thus if we let $\mathcal{I} = \{p(t) \in \mathbb{F}[t] \mid p(t) \text{ monic irreducible}\}$, then we have

$$\beta = \{t^k \mid k \in \mathbb{Z}_{\geq 0}\} \cup \left\{ \frac{t^k}{p(t)^\ell} \;\middle|\; p(t) \in \mathcal{I} \text{ and } k, \ell \in \mathbb{Z} \text{ where } 0 \leq k < \ell \right\}$$

is a basis for $\mathbb{F}(t)$. One can show (roughly because finite subsets of an infinite set have the same cardinality as that set) the cardinality of $\mathbb{F}[t]$ is $\aleph_0 \cdot |\mathbb{F}|$ (which is $|\mathbb{F}|$ if for infinite fields and $\aleph_0$ for finite fields). Note $\mathcal{I}$ contains linear monics: $x - c$ for each $c \in \mathbb{F}$, so $|\mathcal{I}| \geq |\mathbb{F}|$. Also, it can be shown that finite fields have irreducibles of all positive degrees, thus $|\mathcal{I}|$ must always be infinite. Therefore, $|\mathcal{I}| \geq \aleph_0 \cdot |\mathbb{F}|$. Next, since $\mathcal{I} \subseteq \mathbb{F}[t]$, we have its cardinality is at most $|\mathbb{F}[t]| = \aleph_0 \cdot |\mathbb{F}|$, so $|\mathcal{I}| = \aleph_0 \cdot |\mathbb{F}|$. Consider $S = \{(k, \ell) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0} \mid k < \ell\}$. This set has at least $|\mathbb{Z}_{\geq 0}| = \aleph_0$ elements and at most $|\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}| = \aleph_0 \cdot \aleph_0 = \aleph_0$ elements, so $|S| = \aleph_0$. Putting this together,

$$|\beta| = |\{t^k \mid k \in \mathbb{Z}_{\geq 0}\}| + |\mathcal{I}| \cdot |S| = \aleph_0 + (\aleph_0 \cdot |\mathbb{F}|) \cdot \aleph_0 = \aleph_0 \cdot |\mathbb{F}|$$

Theorem: The dimension of the space of rational functions with coefficients in a field $\mathbb{F}$ thought of as a vector space over $\mathbb{F}$ is $\dim(\mathbb{F}(t)) = \aleph_0 \cdot |\mathbb{F}|$ (i.e., $\aleph_0$ for finite fields and $|\mathbb{F}|$ otherwise).

In particular, $\beta = \{1\} \cup \left\{ t^k, \frac{1}{(t-r)^k}, \frac{1}{(t^2+bt+c)^k}, \frac{t}{(t^2+bt+c)^k} \;\middle|\; k \in \mathbb{Z}_{>0} \text{ and } r, b, c \in \mathbb{R} \text{ where } b^2 < 4c \right\}$ is a basis for $\mathbb{R}(t)$ and $\dim(\mathbb{R}(t)) = |\mathbb{R}| = \mathfrak{c}$. And $\beta = \{1\} \cup \left\{ t^k, \frac{1}{(t-z)^k} \;\middle|\; k \in \mathbb{Z}_{>0} \text{ and } z \in \mathbb{C} \right\}$ is a basis for $\mathbb{C}(t)$ and $\dim(\mathbb{C}(t)) = |\mathbb{C}| = \mathfrak{c}$.

References

[BC] T. Bradley and W. Cook, "Two Proofs of the Existence and Uniqueness of the Partial Fraction Decomposition", International Mathematical Forum. 7 (2012), no. 31, 1517–1535.
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