Math 5230 # Homework #1 **Due: Wed., Aug. 28th, 2024**

RREF Handout: https://billcookmath.com/courses/math5230-fall2016/math5230-RREF.pdf

#1 RREF First, (1) use Gaussian elimination (as defined on the handout) to compute the RREF of A. Then, (2) give a vector version of the general solution of $A\mathbf{x} = \mathbf{0}$. Finally, (3) suppose $A = [C \ : \ \mathbf{b}]$ (C is all but the last column and $\mathbf{b}$ is the last column of A). Give a vector form of the general solution of $C\mathbf{x} = \mathbf{b}$.

$$\text{(a)} \quad A = \begin{bmatrix} 1 & -2 & 0 & 2 & 3 \\ 2 & -4 & 1 & 5 & 4 \\ -1 & 2 & 1 & -1 & -5 \end{bmatrix} \qquad \text{(b)} \quad A = \begin{bmatrix} 0 & 1 & 3 & 1 \\ 2 & 0 & 4 & 2 \\ 1 & -1 & -1 & 3 \end{bmatrix}$$

#2 PLU and Partial Pivoting First, (1) use partial pivoting to row reduce A (to REF not RREF), and compute a PLU decomposition for A. Then (2) use this decomposition to solve $A\mathbf{x} = \mathbf{b}$.

$$\text{(a)} \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \qquad \text{(b)} \quad A = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 1 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

#3 Matrix Mechanics First, (1) give a formula for a matrix which performs the given list of row operations. For example: Swapping rows 1 and 2 is $E = I - E_{11} - E_{22} + E_{12} + E_{21}$. Then, (2) give a formula for its inverse (for my example $E^{-1} = E = I - E_{11} - E_{22} + E_{12} + E_{21}$). Next, (3) what happens if you multiply a matrix by your E on the right? (my example swaps columns 1 and 2). Finally, (4) give a concrete example to show off this behavior (for my example I could take some 2×2 matrix and show how E swaps its rows and columns when multiplying on the left or right).

(a) Find a matrix E which swaps rows 3 and 4 and then scales row 2 by 5.

(b) Find a matrix E which swaps rows 1 and 4 and then adds -2 times row 2 to row 3.

#4 More Matrix Mechanics Let $B = I - E_{11} + 3E_{22} - E_{33} - E_{99} + E_{39} + E_{93}$. What does BA do to A? What does AB do to A? (Assume that A and B are appropriately sized.

#5 ~~Poof!~~ Proof! First, let's review some notation and terminology...

Let A be an $m \times n$ matrix with (i, j)-entry denoted a_{ij}. Likewise, let B be a $n \times \ell$ matrix with (i, j)-entry denoted b_{ij}. Recall that AB is an $m \times \ell$ matrix whose (i, j)-entry is given by

$$\sum_{k=1}^{n} a_{ik}b_{kj} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

(i.e., the i^{th} row of A dotted with the j^{th} column of B).

We say that a matrix is *lower triangular* if all of its entries below the main diagonal are zero. In other words, A is lower triangular if $a_{ij} = 0$ whenever $i < j$ (down more rows than over in terms columns). Thus the matrix A is lower triangular if

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} & 0 & \cdots & 0 \end{bmatrix}$$

It is *strictly lower triangular* if $a_{ij} = 0$ whenever $i \leq j$ (i.e., lower triangular with zeros on the diagonal as well).

Problem: Let A and B be $n \times n$ (square) lower triangular matrices. Prove that AB is lower triangular.