Math 5230 # Homework #3 **Due: Wed., Sept. 11th, 2024**

#1 A 2×2 Determinant Consider $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Find $\det(A)$ in the following ways...

(a) Use the usual formula.

(b) Use Gaussian elimination (i.e. the algorithm from class – just use a forward pass).

(c) Compute only using the facts that the determinant is
(1) alternating, (2) bilinear (bi $= 2$ for a 2×2 matrix), and (3) sends the identity matrix to 1.
[*Note:* multilinear plus alternating imply skew.
You may use the skew property (switching rows/cols flips the sign) as well.]

#2 A 3×3 Determinant Consider $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 2 & 1 & 2 \end{bmatrix}$. Find $\det(A)$ in the following ways...

(a) Use the 3×3 trick.

(b) Expand along the first row.

(c) Expand along the second column.

(d) Use Gaussian elimination (i.e. the algorithm from class – just use a forward pass)

(e) The same as part (d) except also use full pivoting.

(f) Find the inverse of this matrix using the classical adjoint formula.

#3 Cramer's Rule Solve $\begin{matrix} 3x & + & 2y & + & 2z & = & 2 \\ x & + & 3y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 1 \end{matrix}$ using Cramer's rule.

#4 Reverse Engineering As Seen in Class!

(a) Create a 5×5 matrix that looks intimidating (make it interesting) even though its determinant is easy to compute when using the right row and column expansions. [Also, compute the determinant of your example using those expansions.]

(b) Create a 4×4 matrix whose entries are all non-zero integers and whose inverse is a matrix whose entries are fractions whose denominators are 4's, 2's, or 1's. [At least one entry of the inverse must be a reduced fraction of the form: ?/4. It's ok if some entries of the inverse are zero.]

#5 Linearly Dependent Gives Zero We know that a set of vectors is linearly dependent if and only if at least one vector can be written as a linear combination of the other vectors in that set. In fact from the linear correspondence property (and Gaussian elimination) we know that if some set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly dependent then there is an index j and constants $c_1, \ldots, c_{j-1}$ such that $\mathbf{v}_j = c_1\mathbf{v}_1 + \cdots + c_{j-1}\mathbf{v}_{j-1}$ (j would be correspond to some non-pivot column).

Let $A = [\mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n]$ (here $\mathbf{a}_i$ is the i^{th} column of A). Show that $\det(A) = 0$ if the columns of A are linearly dependent. Do this only using facts about the effects of column operations on determinants.