We will discuss coordinate matrices for linear transformations in more detail later. For now, I will give an example of finding a standard coordinate matrix for a linear transformation. This is done by successively plugging input (standard) basis vectors into the linear transformation. We write these outputs in terms of their (output) standard coordinates. Each of these coordinate vectors then makes up a column of our coordinate matrix.

Example: Consider, $T : P_2 \to \mathbb{R}^{1\times 2}$ defined by $T(ax^2+bx+c) = [a+2b \;\; a-c]$ where $P_2 = \{ax^2+bx+c \mid a,b,c \in \mathbb{R}\}$ (quadratic, linear, and constant polynomials). We use $\alpha = \{1, x, x^2\}$ (note the order) as our standard basis for P_2 and $\beta = \{\mathbf{e}_1 = [1\ 0], \mathbf{e}_2 = [0\ 1]\}$ as our standard basis for $\mathbb{R}^{1\times 2}$.

Notice $T(1) = T(0x^2+0x+1) = [0 \;\; \text{-}1] = 0\mathbf{e}_1 + (-1)\mathbf{e}_2$ whose standard coordinate vector is $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $T(x) = T(0x^2 + 1x+0) = [2 \;\; 0] = 2\mathbf{e}_1 + 0\mathbf{e}_2$ whose standard coordinate vector is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and $T(x^2) = T(1x^2+0x+0) = [1 \;\; 1] = 1\mathbf{e}_1 + 1\mathbf{e}_2$ whose standard coordinate vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus the standard coordinate matrix for T is $\begin{bmatrix} 0 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

This matrix row reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \end{bmatrix}$. This reveals that the first two columns (i.e., $[0 \text{ -}1]^T$ and $[2\ 0]^T$) are the pivot columns and thus give us a basis for the column space of this matrix. These correspond to the outputs $[0 \;\; \text{-}1]$ and $[2 \;\; 0]$. Thus $\{[0 \;\; \text{-}1], [2 \;\; 0]\}$ (the outputs corresponding to the pivot colums) is a basis for the range of T (so, $\text{rank}(T) = 2$ and thus since $\dim(\mathbb{R}^{2\times 1}) = 2$, our map T is onto).

If we solve our homogeneous system: $x_3 = x_3$ (free), $x_1 - x_3 = 0$ so $x_1 = x_3$ and $x_2 + (1/2)x_3 = 0$ so $x_2 = -x_3/2$, then we get $\left\{ \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix} \right\}$ is a basis for the nullspace of our coordinate matrix. Therefore, $\{1 - x/2 + x^2\}$ (the nullspace basis back out of coordinates) is a basis for the kernel of T (so $\text{nullity}(T) = 1 \neq 0$ and thus T is not one-to-one).

#1 Linear Transformation Basics Define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}[x]$ as follows:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+2b+3d)x^2 + (-a-2b+c+d)x + (a+2b+c+7d).$$

Note that T is a linear transformation (you don't need to prove this).

(a) Write down the standard coordinate matrix for T and find its RREF.
[You may truncate the infinitely many rows of zeros.]

Note: $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ together make up the standard basis for $\mathbb{R}^{2\times 2}$. Also, $\{1, x, x^2, \dots\}$ is the standard basis for $\mathbb{R}[x]$.

(b) Find a basis for the kernel and range of T.

(c) What is the nullity and rank of T? Is T 1-1, onto, both, neither?

#2 Trace Let $A \in \mathbb{F}^{n\times n}$ ($\mathbb{F}$ is a field) and let a_{ij} be the (i,j)-entry of A.

Recall $\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$ (the *trace* of A is the sum of its diagonal entries).

(a) Prove that $\text{tr} : \mathbb{F}^{n\times n} \to \mathbb{F}$ is linear.

(b) Identify the kernel, range, nullity, and rank of the trace map.

#3 Linear Let $\mathcal{I}, \mathcal{D} : \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by $\mathcal{I}(f(x)) = \int_0^x f(t)\,dt$ and $\mathcal{D}(f(x)) = f'(x)$.

(a) Show $\mathcal{I}$ is linear. Identify its kernel and range. Is $\mathcal{I}$ one-to-one? onto? an isomorphism?

(b) Show $\mathcal{D}$ is linear. Identify its kernel and range. Is $\mathcal{D}$ one-to-one? onto? an isomorphism?

(c) Describe $\mathcal{I} \circ \mathcal{D}$ and $\mathcal{D} \circ \mathcal{I}$.